

STRINGS AND STATISTICS

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We discuss the possibility of unusual statistics in four dimensional string theory through the use of Aharonov–Bohm like phases. Such phases are closely related to anomaly cancellation in four dimensions and thus can be exactly calculated in terms of the low energy fermion spectrum of the theory. We illustrate this connection by calculating the statistical phases exactly in sample models of heterotic string compactification. We discuss some of the difficulties in applying these ideas to fundamental strings, and mention some possible applications to cosmic string scenarios.

Since its discovery, fractional statistics of point particles in a plane has been extensively studied. The importance of such fractional statistics has been shown in our understanding of the fractional quantum Hall effect [1–4]. In addition, there is now interest in applying these ideas to the study of high T_c superconductivity [5–7]. More recently, there have been thoughts of generalizing unusual statistics to extended objects in other space–time dimensions, and in particular to the case of strings in four dimensions [8,9].

Since the usual proof of the spin-statistics theorem relies heavily on locality, it is not at all clear that the theorem will continue to be true in string theory. Possible violation of the spin-statistics connection or of CPT invariance would seem to be among the few possible testable differences between string theory and ordinary field theory of point particles. It is thus of crucial importance to understand whether these theorems hold in string theory. There is certainly no

concrete evidence that these principles are violated in string theory, and it seems likely that a deep understanding of them will require going beyond the usual formulation of string perturbation theory.

Our purpose in this paper is to discuss at a rather heuristic level the possibility of exotic statistics in string theory. We will describe how statistical phases may be realized for string theories in four dimensions. We do this by relating string statistical phases to the effects of the four-dimensional anomaly cancellation term $\int B \wedge dA$ that arises in certain cosmic or superstring theories. As a result, knowledge of anomaly cancellation in such a string model allows us to calculate the statistical phase in that theory.

Before considering strings, recall how fractional statistics comes about for particles in a plane. Let Q be the configuration space for N identical particles. Then in three or higher spatial dimensions, $\pi_1(Q)$ is just the permutation group, S_N . The N particle wavefunction must form a representation of S_N , which along with the constraints of locality, Lorentz invariance, and unitarity forces the theory to have Bose or Fermi statistics [10]. However, in the case of two spatial dimensions, $\pi_1(Q)$ is the braid group, B_N , instead. In general, representations of B_N are non-abe-

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lian. However, in most "practical" applications only abelian representations have been considered. This still allows for arbitrary phases upon interchange of identical particles and thus the possibility of fractional statistics.

Such fractional statistics can actually be realized in $(2+1)$ dimensions by the use of a topological Chern-Simons interaction. Consider a theory of particles coupled to an abelian $U(1)$ gauge field,

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{4} \mu \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda - e J_\mu A^\mu, \quad (1)$$

where $\mathcal{L}_0$ is the free particle lagrangian and J_μ the particle current. Since the vector field, A_μ , is non-dynamical, it can be integrated out, leaving us with the effective action

$$\begin{aligned} S_{\text{eff}} &= S_0 - \frac{e^2}{2\mu} \frac{1}{4\pi} \epsilon_{\mu\nu\lambda} \int d^3x d^3y J^\mu(x) \frac{(x-y)^\lambda}{|x-y|^3} J^\nu(y) \\ &= S_0 - \frac{e^2}{2\mu} (\text{linking number}), \end{aligned} \quad (2)$$

where we have recognized the integral as the topological linking number between pairs of particles. Since transporting one particle completely around another changes the linking number by one, the wavefunction picks up an overall phase of $e^2/4\mu$ upon particle interchange. This is the fractional statistics phase and can in general take on any value depending on the Chern-Simons coupling, μ . These novel particles are referred to as anyons [11] and can also be viewed as a particle coupled to a fictitious infinitesimal magnetic flux tube. In this picture, it is obvious that the particles acquire fractional phases through the Aharonov-Bohm effect as they go around each other's flux tubes.

Although this way to realize fractional statistics uses a non-dynamical field, we may give the gauge field a standard kinetic term, $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. The only important change in this case is that each particle's flux tube picks up a characteristic size of $1/\mu$ [12,13]. Thus, for distances $r \gg 1/\mu$ we still preserve the standard fractional statistics, but for shorter distances we see instead a complicated system of interacting particles. Giving A_μ dynamics does not change the large distance fractional statistics but does introduce a length scale at which we begin to see the particles as fundamentally bosons or fermions and not anyons.

Just as we are led to consider the braid group for

particles in a plane, more general possibilities also exist for strings in $(3+1)$ dimensions. Balachandran and collaborators have studied the configuration space of two unoriented strings and have constructed a representation demonstrating exotic statistics [8]. Thus it has been shown that unusual statistics for strings is possible, but we now make this more explicit by an actual construction.

Although Balachandran considers a pair of strings in $(3+1)$ dimensions, we will consider the limiting case in which one string can be treated as a point. Alternatively, for cosmic string scenarios, we can simply consider a theory of strings interacting with point particles carrying the appropriate $U(1)$ charges that are discussed later. The reason behind this is that topologically we can generalize the linking of two-point particle world-lines in $(2+1)$ dimensions to the linking of a string world-sheet with a particle world-line in $(3+1)$ dimensions. The linking number in this case turns out to be

$$\frac{1}{4\pi^2} \int d^2 \Sigma_{\mu\nu}(X) \int dY_\lambda \frac{(X-Y)_\sigma}{|X-Y|^4} \epsilon^{\mu\nu\lambda\sigma}, \quad (3)$$

where $d^2 \Sigma_{\mu\nu}(X) = d\sigma d\tau \epsilon^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\nu$ is the string area element and Y_λ describes the particle world-line. The linking of two strings in $(3+1)$ dimensions is degenerate and reduces to this case.

For the case of fundamental strings, it is possible to relate this linking number to terms in the low-energy effective lagrangian which describes the theory after compactification to four dimensions. Upon compactification, we are often led to anomalous $U(1)$ factors in the effective theory. As the underlying ten dimensional theory is anomaly free, there must be a mechanism for anomaly cancellation in four dimensions. It is this mechanism that gives the crucial $\int B dA$ term that leads to the statistical phase. Since the anomaly coefficient depends on the massless fermion spectrum, knowledge of the low-energy spectrum will give us the precise value of this statistical phase in the model of interest. Although we work in four dimensions, the existence of six compactified dimensions or additional degrees of freedom can probably not be ignored and may serve to destroy the topological property of the string linking. Thus these extra dimensions cause problems if we attempt to use this method of generating exotic statistics for superstrings. On the other hand, certain cosmic strings,

such as those discussed in ref. [14], use a similar anomaly cancellation mechanism and can pick up exotic statistics as well without this problem of extra dimensions. We will discuss problems due to compactified dimensions at the end.

To see how this phase comes about, now let us specialize to the case of the heterotic string. We will only concentrate on the bosonic terms in the effective low energy string action. The four dimensional heterotic string modes can be described by a sigma model action [15–18]:

$$S = \frac{1}{2\pi\alpha'} \int d^2z [\sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + \frac{1}{4} \alpha' \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \alpha' \sqrt{h} \phi R^{(2)}(h) + 2\alpha' \bar{\psi}_L^A \partial_+ \psi_L^A + 2\alpha' J_L^a A_\mu^a \partial_+ X^\mu], \quad (4)$$

where $\mu, \nu = 1, \dots, 4$ are space-time indices and $\alpha, \beta = 1, 2$ world-sheet ones. Here, we have used the fermionic formulation for the left moving gauge degrees of freedom: $\psi_L^A, A = 1, \dots, 32$ generates either the $SO(32)$ or the $E_8 \times E_8$ theory, and $J_L^a = \bar{\psi}_L t^a \psi_L$ with t^a the group generators normalized such that $\text{tr}(t^a t^b) = \delta^{ab}$. The coefficients of the various terms have been chosen so that the sigma model action describes the same dynamics as the four dimensional super-gravity super-Yang–Mills action [19–21] with standard normalization which is the low-energy field-theory limit of string theory [22]. A_μ^a is the Yang–Mills gauge field and $B_{\mu\nu}$ the antisymmetric tensor field.

Now we must take into account anomaly cancellation in four dimensions. In general, we may diagonalize the potentially anomalous $U(1)$ factors, leaving us with only a single anomalous $U(1)$. Writing the resulting broken gauge group as $U(1) \otimes G$ with A the $U(1)$ gauge field, the anomaly under the A gauge transformation, $\delta_A A = DA$, is given by [23, 14]

$$\delta_A S = -\frac{c}{32\pi^2} \int (F^2 + \text{tr} G^2 - \text{tr} R^2) A, \quad (5)$$

where F is the $U(1)$ field strength, G that of the remaining gauge group, and R the curvature tensor. Here, we are using the notation and normalization of ref. [14]. On grounds of modular invariance, all three terms contribute with the same coefficient which is determined through the massless fermion content

$$c = \frac{1}{3} \sum_i a_i^3 \dim r_i = \sum_i a_i \text{tr} \lambda_i^2 = \frac{1}{24} \sum_i a_i \dim r_i, \quad (6)$$

where a_i is the $U(1)$ charge of the fermion in the r_i representation of G and λ_i is the corresponding representation matrix. It is this equality that allows the anomaly to be cancelled by the addition of the term [24]

$$\frac{c}{32\pi^2} \int BF = \frac{c}{32\pi^2} \int d^4x \epsilon_{\mu\nu\lambda\sigma} B^{\mu\nu} \partial^\lambda A^\sigma \quad (7)$$

to the effective action where $B_{\mu\nu}$ has the usual transformation property $\delta_A B = \omega_{\frac{1}{2}A}$. This anomaly cancellation is similar to the Green–Schwarz mechanism [25] and can be shown to come from higher order string effects.

Looking at the terms of interest, we have

$$S' = \frac{1}{8\pi} \int d^2z \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \frac{1}{\pi} \int d^2z A_\mu \partial_+ X^\mu J_L + \frac{c}{32\pi^2} \int B dA, \quad (8)$$

where in this case J_L is the anomalous $U(1)$ world-sheet gauge current and A_μ the $U(1)$ gauge field. Assuming a small extent for the string coupled to the gauge current, we may integrate out the antiholomorphic coordinate to get

$$\frac{1}{\pi} \int d^2z A_\mu(X) \partial_+ X^\mu J_L \approx Q \int dl A_\mu(Y) \frac{\partial Y^\mu}{\partial l}, \quad (9)$$

where $Q = (1/\pi) \int d\sigma J_L(\sigma)$ is the anomalous $U(1)$ charge of the string in its particular mode. Here, $Y^\mu(l) \approx X^\mu(z, \bar{z})$ describes the world-line of the infinitesimal string. Note that in a theory including point particles, we may simply take A_μ coupled to the particle current $\partial Y^\mu / \partial l$. Thus we are led to

$$S' = \frac{1}{8\pi} \int d^2z \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + Q \int dl A_\mu \frac{\partial Y^\mu}{\partial l} + \frac{c}{32\pi^2} \int B dA, \quad (10)$$

with Q the $U(1)$ charge of the infinitesimal string, c is the four dimensional anomaly coefficient. Now, ignoring other terms and in particular dynamical terms

for A_μ and $B_{\mu\nu}$, we may simply integrate out these fields to get [26]

$$S'_{\text{eff}} = \frac{4\pi Q}{c} \frac{1}{4\pi^2} \int d^2 \Sigma_{\mu\nu} \int dY_\lambda \frac{(X-Y)_\sigma}{|X-Y|^4} \epsilon^{\mu\nu\lambda\sigma}, \quad (11)$$

which we recognize as the string/particle linking number:

$$S'_{\text{eff}} = \frac{4\pi Q}{c} (\text{linking number}). \quad (12)$$

This linking number can be thought of physically as the number of times the particle goes through the string. If this is not a multiple of 2π , we pick up a statistical phase for different linkings. In order to connect this to exotic statistics, recall that the string Hilbert space must give a representation of the fundamental group of the configuration space. As $\pi_1(Q)$ is in this case more complicated than S_N , we may expect different types of interactions for strings than for particles [8]. In particular, there are two possible types of string exchange, one with zero linking number, and one with unit linking number where one string goes through the middle of the other (called a slide in ref. [8]). It is this term that picks up the above calculated statistical phase.

This method of giving Aharonov–Bohm like phases to strings is similar to that of fractional statistics in a plane. However, one major difference is that in the plane anyons must pick up a fixed statistical phase upon interchange. For the case of string interchange, we may or may not pick up a phase depending on how the exchange is performed. Ordinary string exchange in this model will not pick up any phases whereas a slide will. As a result, the interpretation of these unusual string statistics is more subtle.

As an explicit example of how this phase arises, first consider the case of the $SO(32)$ heterotic string in Calabi–Yau compactification [27,28]. As $SO(32) \supset SU(3) \times U(1) \times SO(26)$, we can break the gauge group down to $SO(26) \times U(1)$ by embedding the spin connection in the gauge group. It is this $U(1)$ factor that turns out to be anomalous. By examining the massless fermion spectrum, we find the $U(1)$ anomaly coefficient to be $c = -\frac{3}{2}\chi$ with χ the Euler characteristic of the compactified space. With this particular normalization, the fermions' $U(1)$ charges are ± 1

or ± 2 which leads to the statistical term

$$S'_{\text{eff}} = -\frac{8\pi Q}{3\chi} (\text{linking number}),$$

$$Q = \pm 1, \pm 2. \quad (13)$$

As this is not a multiple of 2π , the $SO(26) \times U(1)$ theory has potentially unusual statistics.

For another example, consider the $SU(3) \times SU(2)$ orbifold model of Ibáñez et al. [29]. In this model, the $E_8 \times E_8$ gauge group is broken down to $[SU(3) \times SU(2) \times U(1)^5] \times [SU(2) \times SU(2) \times U(1)^6]$ in an orbifold compactification with two Wilson lines. In this case, of the 11 potentially anomalous $U(1)$ factors, it has been shown that only one linear combination is truly anomalous [30]. Looking at the fermion spectrum gives us the anomaly coefficient $c = \frac{1}{2}\sqrt{21}$. The anomalous $U(1)$ charge is normalized to $Q = \frac{1}{2}n/\sqrt{21}$, $n \in \mathbb{Z}$, and this gives a statistical phase of

$$S'_{\text{eff}} = \frac{4}{21}\pi n (\text{linking number}), \quad (14)$$

which again would contribute unusual statistics.

As these two examples show, many compactification schemes would in general lead to a non-trivial statistical phase. The physical consequences of this phase are not completely clear however. So far we have been working in an approximation where we ignore both the kinetic energy terms for the gauge and antisymmetric tensor fields and the presence of extra dimensions in string theory. In string compactifications with a geometrical interpretation there seems to be a minimal size of the compactified space of order $\sqrt{\alpha'}$ which implies that there are always excitations of the internal space with energy of order $1/\sqrt{\alpha'}$. In addition, as for particles in a plane, adding the four dimensional kinetic terms $-(1/4g^2)F_{\mu\nu}F^{\mu\nu} - (3\alpha'/32g^2)H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ where $H = dB - \omega_3 = dB - A \wedge dA$ introduces a length scale of order $\lambda \approx 1/cM_s \equiv \sqrt{\alpha'}/c$ with c the anomaly coefficient and M_s the string scale. Thus we expect generalized “flux tubes” with size of order of the string scale around the strings.

As long as we are considering the interaction of point particles with macroscopic strings of size much greater than the string scale we should be able to ignore both of these effects. The phase discussed here will then act in much the same way as the phase that arises through the ordinary Aharonov–Bohm effect,

although it arises in quite a different way. This can lead to enhanced scattering of particles off of cosmic strings as discussed in ref. [31].

A more interesting and problematic possibility is that these phases could lead to corrections to the spin-statistics relation for point particles in four dimensions. If we ignore the extra dimensions and the finite size of the "flux tube" around the string, then we might expect such corrections to arise from virtual exchange processes in which one of the particles opens into a closed string which the other particle then passes through. We can make a rough estimate for the action for such a process by calculating the area of a string world-sheet which starts as a point, opens up to a size of order of the Compton wavelength of the other particle, and then shrinks back to a point. We thus expect such processes to be suppressed by a factor $\exp(-M_s^2/E^2)$, where M_s is the string scale and E the particle energy. Although unobservably small, any violation of the spin-statistics relation would be of interest to study and understand.

It is clear, however, that the extra dimensions and kinetic energy terms can not really be ignored in a calculation of such an effect. In particular, at some level the presence of extra dimensions will destroy the purely topological nature of the linking number. It is of course now possible to formulate string theories without direct reference to extra dimensions and in some constructions, such as asymmetric orbifold constructions, there is no clear geometrical interpretation in terms of a compactified space. Clearly one needs a more fundamental string calculation of the possible correction to spin-statistics in order to properly include the additional degrees of freedom present in string theory. We have so far been unable to formulate such a direct calculation.

In the above, we have demonstrated how anomaly cancellation in four dimensions can lead to non-trivial statistical phases and potentially interesting string-statistics. Even though there are problems in interpreting this phase in the case of compactified fundamental strings, we still hope the general connection between spin and statistics of four dimensional strings may be understood. As was mentioned previously, the more complicated fundamental group of the string configuration space leads to an enlarged Hilbert space for four dimensional strings. It may be possible to study the statistics of strings without any detailed cal-

culation if this enlarged Hilbert space can somehow be better understood. If so, this could allow us to study phenomena unique to strings that may provide an interesting distinction between ordinary point particle theories and string theories.

The connection between statistics of strings and the string analog of the Chern-Simons coupling is also discussed in ref. [32].

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