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CONFORMAL FIELD THEORY AND GENUS-ZERO FUNCTION FIELDS[1]

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One-loop partition functions of rational conformal field theories are finite linear combinations of modular invariants associated with projective modular functions of a modular subgroup. We show that, for normal subgroups with a genus-zero fundamental region, the functions which lead to physically acceptable partition functions are extremely limited in number, and can be found explicitly. We also show that the conformal charge and weights of theories which factorize on these subgroups can only take on certain discrete values.

1. Introduction

Two-dimensional conformal field theories constitute classical string vacua, so a greater understanding of the moduli space of these theories would aid in the construction of string compactifications. In the context of string theory, conformal theories live on the string world sheet, and therefore we must be able to define them consistently not only on the sphere but on higher genera Riemann surfaces as well. For the theory to be viable, its higher loop partition and correlation functions must be free of global reparametrization or modular anomalies. In particular, the one-loop partition function must be modular invariant, which imposes stringent conditions on the spectrum of the theory.

For conformal charge $c < 1$, the combination of conformal invariance, unitarity and one-loop modular invariance is powerful enough to restrict the spectrum of the conformal theory to a small number of possible choices [1–3]. Null states in the Virasoro representations lead to differential equations that the correlation functions must satisfy, providing constraints on the operator product algebra [1]. Moreover, all of this can be deduced without knowing anything about the nature of the conformal theory with which one is dealing.

The moduli space of conformal field theories for $c \geqslant 1$ is more complicated, and often contains continuous spaces as well as isolated points. To make progress in understanding these theories, it seems necessary to add some further restrictions.

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One circle of ideas on how this might be done centers around the notion of a rational conformal field theory. Although there are several possible definitions of a rational conformal field theory [4–6], in this paper we will take it to mean a conformal field theory whose one-loop partition function has the form

$$Z(\tau,\bar{\tau}) = \sum_{i=1}^{M} f_i(\tau)\bar{g}_i(\bar{\tau}), \tag{1.1}$$

where f_i and g_i are holomorphic in the upper half plane, M is finite, and Z is modular invariant. In fact, we will strengthen this definition by requiring that Z factorize on some finite index subgroup Γ' of the modular group, i.e., that f_i and g_i are modular functions (perhaps with multiplier systems) of $\Gamma' \subseteq \Gamma$ [5]. Many of the known conformal field theories are rational in this sense: the minimal series of $c < 1$ theories [1, 2], Wess–Zumino–Witten models [7], the bosonic string compactified on a rational lattice [8, 9].

Since rational lattices are dense in the space of all lattices, it is clear that rational conformal field theories are "dense" in the space of all conformal field theories with a toroidal target space. There is evidence that there exist irrational conformal field theories which are not "close" to any rational conformal field theory [5], so that we are not likely to be able to obtain *all* conformal field theories by completing the space of rational conformal field theories. Nonetheless, we may find a large and interesting class of them.

In this paper, we examine certain subgroups of the modular group and show how to classify all one-loop partition functions which factorize on these groups and which satisfy certain physical requirements described below. In particular, we investigate subgroups whose fundamental regions have genus zero, because all modular functions (with rational multiplier systems) of these subgroups are algebraic functions of a single function, called the *hauptfunktion*, of the subgroup. From a pair of such functions, f and g, which have the same multiplier systems, we construct the modular invariant $\mathscr{Y}(f, g)$ as a sum over coset transformations

$$\mathscr{Y}(f,g) = \sum_{k=1}^{\mu} f(A_k\tau)\bar{g}(A_k\bar{\tau}), \tag{1.2}$$

where $\Gamma = \bigcup_{k=1}^{\mu} \Gamma' A_k$ is a coset decomposition of Γ' in Γ, and μ is the finite index $(\Gamma : \Gamma')$. Any partition function (1.1) which factorizes on Γ' will be a finite linear combination of the $\mathscr{Y}$'s. In addition, we impose on Z the following physical requirements.

(R_1) The functions f_i and g_i must be holomorphic (no poles or branch points) in the upper half plane.

(R_2) In the cusp expansion of the partition function

$$Z = \mathrm{Tr}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} = q^{-c/24} \bar{q}^{-c/24} \sum_{i,j} N_{ij} q^{h_i} \bar{q}^{\bar{h}_j}, \tag{1.3}$$

where $q = e^{2\pi i \tau}$ is the local parameter at the cusp $\tau = i\infty$, and the conformal weights $h_i, \bar{h}_j$ are non-negative, we require the coefficient for $h = \bar{h} = 0$ to be one, because the theory must have exactly one identity operator.

(R_3) With the normalization of Z prescribed in R_2, the coefficients N_{ij} must be integers, since they correspond to the multiplicity of states with conformal weights $h_i, \bar{h}_j$.

(R_4) Unitarity requires the coefficients N_{ij} to be non-negative.

(R_5) Actually, R_4 must be strengthened. Conformal invariance allows us to write the partition function as

$$Z = \sum_{i,j} M_{ij} \chi_{h_i} \bar{\chi}_{\bar{h}_j}, \tag{1.4}$$

where χ_h is the character of the irreducible representation of the Virasoro algebra with highest weight h. Unitarity then requires the coefficients M_{ij} to be non-negative integers. For $c > 1$, the Virasoro characters with $h > 0$ have the form

$$\chi_h(q) = \frac{q^{h-(c-1)/24}}{\eta(\tau)}, \tag{1.5}$$

where $\eta(\tau)$ is the Dedekind eta function $q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, whereas the character of vanishing highest weight is $\chi_0(q) = q^{-(c-1)/24}(1 - q)/\eta(\tau)$. For $c = 1$ and $h \neq \frac{1}{4}n^2$, $n \in \mathbb{Z}$, eq. (1.5) still holds, but for $h = \frac{1}{4}n^2$ the characters are

$$\chi_{n^2/4}(q) = \frac{q^{n^2/4} - q^{(n+2)^2/4}}{\eta(\tau)} = \frac{q^{n^2/4}(1 - q^{n+1})}{\eta(\tau)}. \tag{1.6}$$

Because of the existence of null states in some of the representations, this stronger form of unitarity is a little difficult to check. R_5 certainly implies R_4, so that we will invoke it only after a candidate partition function has passed all the other tests.

(R_6) Since the conformal field theory must exist on an arbitrary Riemann surface, an allowable one-loop partition function must admit a higher loop generalization which is also modular invariant. This is known to provide additional constraints on the one-loop partition function [10]. In this paper we will make no use of this further requirement.

In sects. 2 and 3, we review some facts about the modular group and its subgroups, their fundamental regions, and the modular functions defined on these

fundamental regions. In sect. 4, we determine the partition functions that factorize on the full modular group. In sect. 5, we discuss how to characterize partition functions that factorize on genus-zero subgroups. In sects. 6 and 7, we apply these results concretely to the subgroups $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$ and $\Gamma(5)$, deriving restrictions on c and h and describing the modular invariants which comprise the partition functions which factorize on these subgroups. In sect. 8, we discuss possible extensions of this work and present our conclusions.

2. The modular group

In the next two sections, we review some standard material from the theory of the modular group. Much of it is taken from refs. [11–15].

The one-loop string world sheet has the topology of a torus, which can be represented by the complex plane modded out by a lattice generated by two basis vectors (complex numbers) ω_1 and ω_2. Without loss of generality, we require $\mathrm{Im}(\omega_2/\omega_1) > 0$. The same lattice is generated by two new basis vectors

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \tag{2.1}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. A function $f(\omega_1, \omega_2)$ of the torus should not be sensitive to the arbitrary choice of basis vectors, i.e., should be invariant under the $\mathrm{SL}(2,\mathbb{Z})$ transformation (2.1). Conformal invariance allows us to scale ω_1 to 1 and ω_2 to $\tau \in \mathscr{H}$, the upper half plane. The transformation (2.1) corresponds to

$$\tau' = \frac{a\tau + b}{c\tau + d}, \tag{2.2}$$

and these transformations of the modular parameter τ constitute the (inhomogeneous) modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm \mathrm{I}\}$, in which $A \in \mathrm{SL}(2,\mathbb{Z})$ is identified with $-A$ because both have the same action on τ. Γ is generated by the two transformations S: $\tau \to -1/\tau$ and T: $\tau \to \tau + 1$, which satisfy the relations $S^2 = (ST)^3 = 1$.

A fundamental region for Γ is a subset $\mathscr{F}$ of $\mathscr{H}$ containing exactly one point from each class of points equivalent under eq. (2.2). A standard choice, shown in fig. 1, is

$$\mathscr{F} = \left\{ \tau \in \mathscr{H} \,|\, -\tfrac{1}{2} \leqslant \mathrm{Re}\,\tau \leqslant \tfrac{1}{2}, |\tau| \geqslant 1 \right\} \cup \{ i\infty \}, \tag{2.3}$$

in which we have included the point $\tau = i\infty$. This point and the points Γ-equivalent to it, viz., all rational points on the real axis, are called cusps. In looking at fig. 1, we are to identify the line $\mathrm{Re}\,\tau = -\frac{1}{2}$ with $\mathrm{Re}\,\tau = \frac{1}{2}$, and the left half of the arc of the

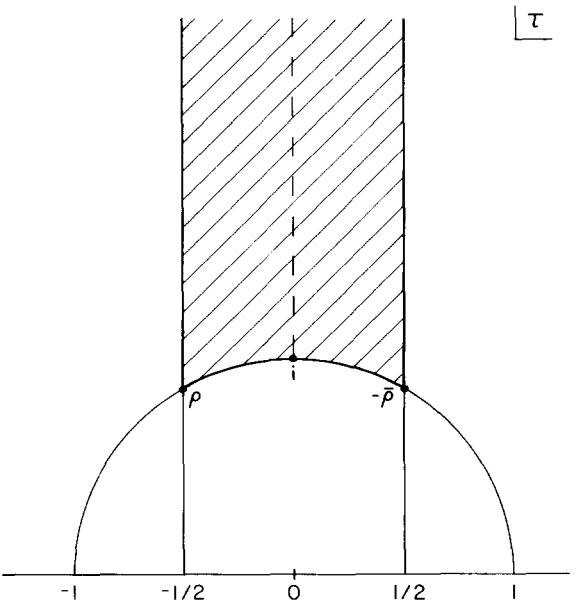

Fig. 1. $\mathcal{F}$, the fundamental region for Γ.

unit circle with the right half. By folding over the strip, crimping the bottom end and pinching off the cusp, we see that $\mathcal{F}$ has the topology of a sphere. The points of $\mathcal{F}$ are in one-to-one correspondence with the points of the compactification $\widehat{\mathcal{H}/\Gamma}$ of $\mathcal{H}/\Gamma$. Moreover, $\widehat{\mathcal{H}/\Gamma}$ can be given a complex analytic structure, so we are to think of $\mathcal{F}$ as a Riemann surface of genus zero, i.e., the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

In order to deal with functions living on $\mathcal{F}$, we must understand the nature of some special points belonging to it. If we circle the point $\tau = i$ once, using the identifications mentioned above, we only go through an angle π, hence i is an "orbifold" point of order 2. Similarly, $\tau = \rho = e^{2\pi i/3}$ is an orbifold point of order 3. Even more special is the cusp $\tau = i\infty$, for which we define a local variable $q = e^{2\pi i\tau}$, which maps the top part of the strip into the interior of a disc. Thus, by crossing the strip, we "circle" the cusp. We will call a function meromorphic (holomorphic) at $i\infty$ if it possesses a Laurent expansion in q with a largest negative power (with no poles).

A modular form (an entire modular form) of weight k is a function $f(\tau)$ meromorphic (holomorphic) in the upper half plane and at $i\infty$ which satisfies

$$f(A\tau) = (c\tau + d)^k f(\tau), \qquad A\tau = \frac{a\tau + b}{c\tau + d}, \qquad A \in \Gamma. \tag{2.4}$$

A modular function is a modular form of zero weight, i.e., is modular invariant and thus well defined on $\widehat{\mathcal{H}/\Gamma} \sim \mathcal{F}$. As is well known, the sum of the orders of zeros equals the sum of the orders of poles for meromorphic functions on a Riemann surface. For modular functions on $\mathcal{F}$, the counting is slightly modified at the orbifold points, and we have

$$v_{i\infty}(f) + \tfrac{1}{2}v_i(f) + \tfrac{1}{3}v_\rho(f) + \sum_{\substack{\tau\in\mathcal{F}\\ \tau\neq i,\,\rho,\,i\infty}} v_\tau(f) = 0, \tag{2.5}$$

where $v_\sigma(f)$ is the order of f at σ, i.e., the integer n such that $f(\tau)/(\tau-\sigma)^n$ is holomorphic and non-zero at σ. (The order of f at $i\infty$ is the leading power of q in its Laurent expansion.) For modular forms of weight k, the right-hand side of eq. (2.5) is replaced by $\frac{1}{12}k$.

All modular forms may be expressed in terms of the Eisenstein series

$$G_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}, \qquad \tilde{G}_k(\tau) = \frac{1}{2\zeta(k)}G_k(\tau), \tag{2.6}$$

where $\tilde{G}_k$ is defined so that $\tilde{G}_k(i\infty) = 1$. One may easily verify that the series, which converges absolutely for $k > 2$, defines for $k \in 2\mathbb{Z}$ an entire modular form of weight k. (It vanishes for odd k.) Products of modular forms are likewise modular forms, and in fact, $\tilde{G}_4$ and $\tilde{G}_6$ generate the graded ring of entire modular forms. Twelve is the lowest weight for which there are two independent entire modular forms, $\tilde{G}_4^3$ and $\tilde{G}_6^2$, and we define a linear combination of them which vanishes at the cusp but nowhere else, called the discriminant, $\Delta(\tau) = (2\pi)^{12}(\tilde{G}_4^3 - \tilde{G}_6^2)/1728 = (2\pi)^{12}\eta^{24}(\tau)$ $= (2\pi)^{12}q\prod_{n=1}^{\infty}(1-q^n)^{24}$. This enables us to define the modular function

$$j(\tau) = \frac{\tilde{G}_4^3}{\eta^{24}} = 1728 + \frac{\tilde{G}_6^2}{\eta^{24}}, \tag{2.7}$$

which is manifestly holomorphic in the upper half plane and has a simple pole at the cusp. $\tilde{G}_4$ vanishes at $\tau = \rho$ as is easily shown by the following string of identities

$$\tilde{G}_4(\rho) = \tilde{G}_4(\rho+1) = \tilde{G}_4(-\bar{\rho}) = \frac{1}{2\zeta(4)}\sum\frac{1}{\bar{\rho}^4(-m+n\rho)^4} = \frac{1}{\bar{\rho}}\tilde{G}_4(\rho) = 0, \tag{2.8}$$

where the first equality holds by virtue of eq. (2.4), the second by $\rho + \bar{\rho} + 1 = 0$, the third by eq. (2.6) and by $\bar{\rho}\rho = 1$, and the fourth by a change of summation variables.

Similarly, $\tilde{G}_6$ vanishes at $\tau = i$

$$\tilde{G}_6(i) = \frac{1}{2\zeta(6)} \sum \frac{1}{i^6(m-in)^6} = -\tilde{G}_6(i) = 0, \tag{2.9}$$

so that by eq. (2.7) we have $j(\rho) = j(i) - 1728 = 0$. The function j takes every complex value once on $\mathscr{F}$, and thus defines an isomorphism of $\widetilde{\mathscr{H}/\Gamma}$ onto $\hat{\mathbf{C}}$. Finally, just as every meromorphic function on $\hat{\mathbf{C}}$ is a rational function of z (i.e., ratio of polynomials in z with complex coefficients), so too is every modular function a rational function of j

$$f \propto \frac{\prod_{j=1}^{n}(j - a_j)}{\prod_{k=1}^{m}(j - b_k)}. \tag{2.10}$$

3. Modular subgroups

Let Γ' be a subgroup of Γ with finite index $\mu = (\Gamma : \Gamma')$. If $A_k \in \Gamma$, $k = 1, \ldots, \mu$, is a set of coset representatives of Γ' in Γ, i.e., $\Gamma = \bigcup_{k=1}^{\mu} \Gamma' A_k$, then $\mathscr{F}' = \bigcup_{k=1}^{\mu} A_k(\mathscr{F})$ is a fundamental region for Γ'. The principal congruence subgroups, which are labeled by their level, N, are defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, a, d = 1 \bmod N;\ b, c = 0 \bmod N \right\}. \tag{3.1}$$

These are normal subgroups of Γ and have index

$$\mu(N) = \begin{cases} 6, & N = 2, \\ \frac{1}{2}N^3 \prod_{p|N} (1 - 1/p^2), & N > 2, \end{cases} \tag{3.2}$$

where the product is over all primes p which divide N. We have listed the indices and some other pertinent properties of the low level $\Gamma(N)$'s in table 1 for convenience. We refer to the fundamental region for $\Gamma(N)$ as $\mathscr{F}_N$. $\mathscr{F}_2$ and $\mathscr{F}_4$ are

TABLE 1
Indices, number of cusps and genera for $\Gamma(N)$, $N \leqslant 12$

N	1	2	3	4	5	6	7	8	9	10	11	12
μ	1	6	12	24	60	72	168	192	324	360	660	576
σ_∞	1	3	4	6	12	12	24	24	36	36	60	48
g	0	0	0	0	0	1	3	5	10	13	26	25

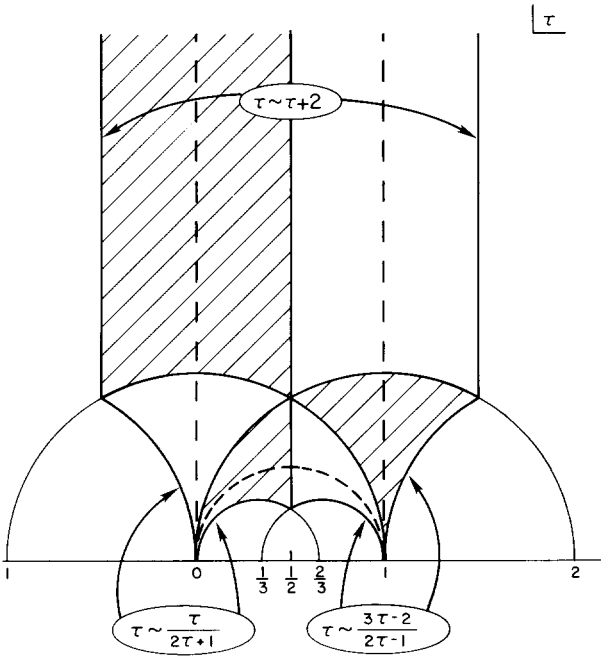

Fig. 2. $\mathscr{F}_2$, the fundamental region for $\Gamma(2)$.

shown in figs. 2 and 3, in which the arrows indicate which edges of the boundaries are identified under $\Gamma(N)$. It is not hard to convince oneself, by pulling together the equivalent edges, that $\mathscr{F}_2$ and $\mathscr{F}_4$, like $\mathscr{F}$, are topologically spheres. For brevity, we will call such subgroups genus-zero subgroups.

Following ref. [11], we now derive a formula for the genus of the fundamental region of any subgroup Γ'. $\mathscr{F}'$ is composed of μ modular triangles, which are Γ-transforms of $\mathscr{F}$. We divide these into half triangles by joining the cusp to the point Γ-equivalent to i with a hyperbolic straight line (the dotted lines in fig. 2). Let V_i, $i = 1,\ldots,v$, be the vertices of $\mathscr{F}'$ and let $2\kappa_i$ be the number of half triangles touching V_i. The division of $\mathscr{F}'$ into half triangles constitutes a polygonal decomposition, with v vertices, 2μ faces, and $\sum_{i=1}^{v}\kappa_i$ edges, whence its genus is

$$g = -\tfrac{1}{2}(v - e + f - 2) = \tfrac{1}{2}\sum_{i=1}^{v}(\kappa_i - 1) - \mu + 1. \tag{3.3}$$

In this paper, we will focus our attention on normal subgroups because they are particularly easy to analyze. Henceforth, we assume that Γ' is normal. In that case, there are equally many half triangles touching at vertices which are Γ-equivalent to i, ρ and $i\infty$, which we denote by $2n_i$, $2n_\rho$ and $2n_\infty$, respectively. The triple

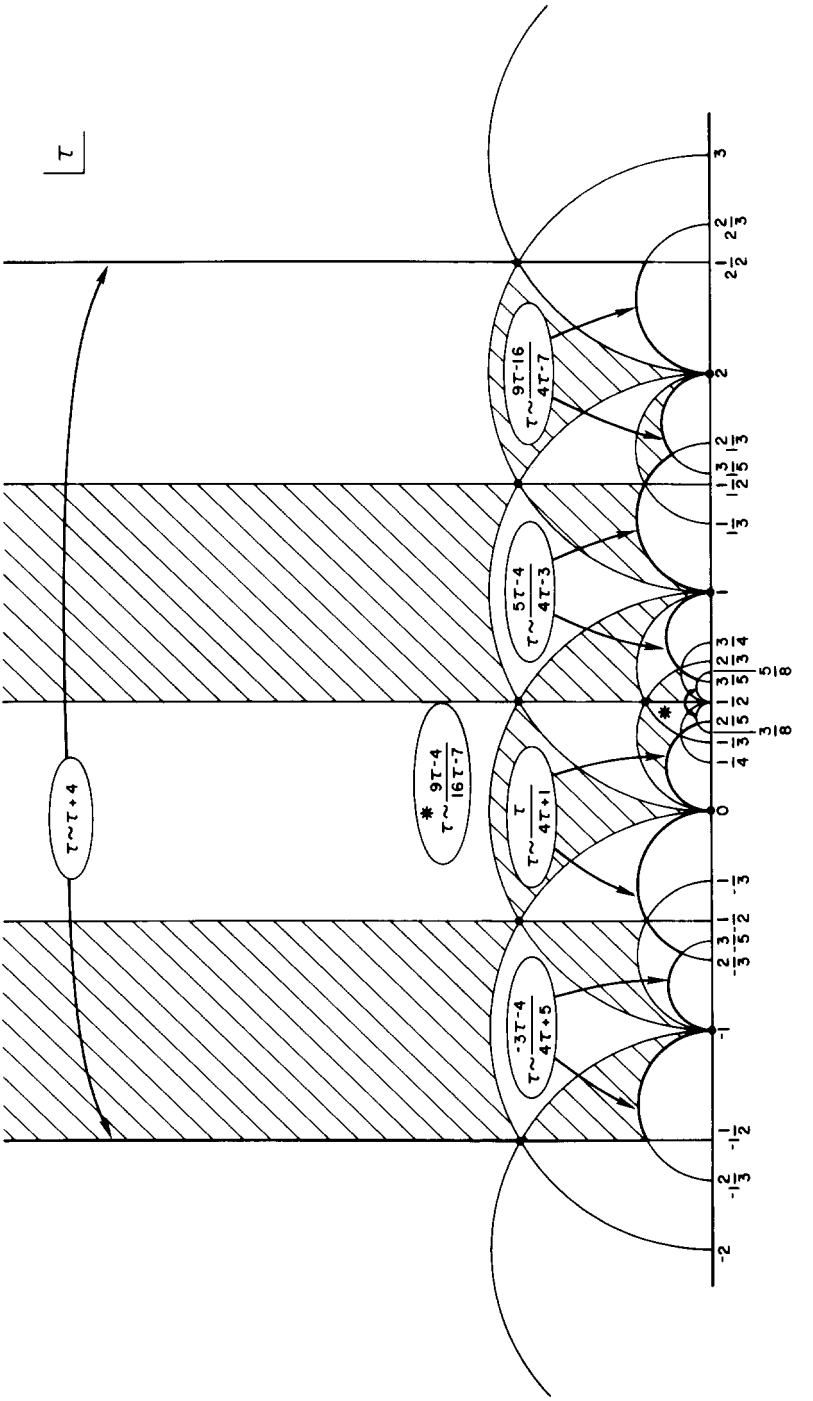

Fig. 3. $\mathscr{F}_4$, the fundamental region for $\Gamma(4)$.

(n_i, n_ρ, n_∞) is called the branch schema of the normal subgroup. Since the numbers of vertices Γ-equivalent to i, ρ and $i\infty$ are μ/n_i, μ/n_ρ and μ/n_∞, we obtain for the genus

$$g = \frac{\mu}{n_i}\frac{n_i - 1}{2} + \frac{\mu}{n_\rho}\frac{n_\rho - 1}{2} + \frac{\mu}{n_\infty}\frac{n_\infty - 1}{2} - \mu + 1 . \tag{3.4}$$

The number of modular triangles fanning out from each cusp, n_∞, is called the fan width, and for $\mathcal{F}_N$ is equal to the level N. The number of cusps, $\sigma_\infty = \mu/n_\infty$, of $\mathcal{F}_N$ is thus μ/N. Clearly, for any subgroup, $n_i = 1$ or 2, $n_\rho = 1$ or 3, and $g \geqslant 0$, so that the branch schema must assume one of the following forms.

$(n_i, n_\rho, n_\infty) = (1,1,1)$. This is the branch schema of the full modular group.

$(n_i, n_\rho, n_\infty) = (2,1,2)$. This is the branch schema of a genus-zero subgroup of index 2, denoted Γ^2, and generated by ST and ST^{-1}.

$(n_i, n_\rho, n_\infty) = (1,3,3)$. This is the branch schema of Γ^3, with index 3 and genus zero, whose generators are S, $T^{-1}ST$ and $T^{-1}ST^{-2}S$.

$(n_i, n_\rho, n_\infty) = (2,3,N)$, $N \geqslant 2$. These are the branch schemata for the principal congruence subgroups, although for some N there are other subgroups with the same branch schema. For example, $\Gamma(6)$ and Γ_C, the commutator subgroup of Γ, share the branch schema (2,3,6). The genus formula (3.4) for $(n_i, n_\rho, n_\infty) = (2,3,N)$ reduces to

$$g = \frac{\mu(N)}{N}\frac{N-6}{12} + 1 . \tag{3.5}$$

Specifically, we note that $g = 0$ for $N = 2$, 3, 4 and 5. Furthermore, for these values of N, the principal congruence subgroups are the only subgroups with this branch schema. Hence, there are exactly 6 normal genus-zero proper subgroups.

Modular functions for a subgroup Γ' are meromorphic in the upper half plane and at the cusps, and satisfy $f(A\tau) = f(\tau)$ for all $A \in \Gamma'$. Thus, they are well defined on $\widetilde{\mathcal{H}/\Gamma'} \sim \mathcal{F}'$, which can be given a complex analytic structure. For genus-zero subgroups, there exists a modular function, called the hauptfunktion h, which maps $\mathcal{F}'$ one-to-one to the Riemann sphere $\hat{\mathbb{C}}$. All modular functions for Γ' can be written as rational functions of h, which is the analog of j for Γ'.

A modular function for the principal congruence subgroup $\Gamma(N)$ is called a modular function of level N. We denote the hauptfunktions for $\Gamma(N)$, $2 \leqslant N \leqslant 5$, by h_N. The easiest way to construct h_N explicitly is as a ratio of modular forms of level N, which we now do for h_2, h_3 and h_4, and indicate how to do for h_5.

(1) h_2. The three Jacobi theta functions, ϑ_2, ϑ_3 and ϑ_4, transform into each other under modular transformations, but under $\Gamma(2)$ transformations, each goes into itself as a form of weight $\frac{1}{2}$, up to a phase (see appendix A). If we raise them to the fourth power, the phase disappears and so the ϑ_i^4 are entire modular forms of weight 2 and level 2. Only two of these are independent by virtue of Jacobi's

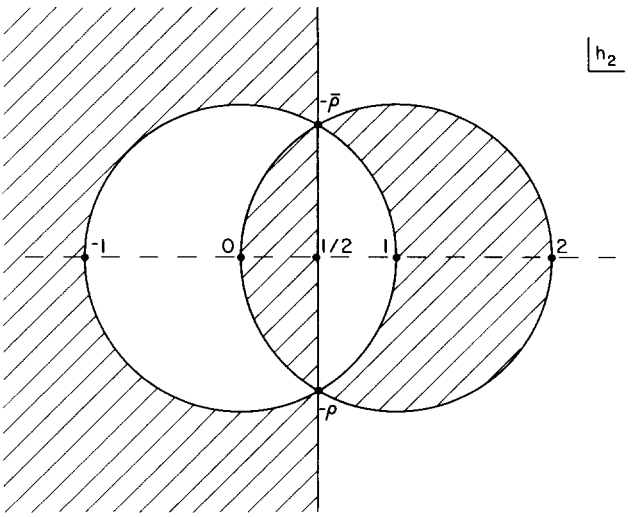

Fig. 4. The image of the map h_2: $\mathscr{F}_2 \to \hat{\mathbb{C}}$.

identity, $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$, and from these we define the level-2 function

$$h_2(\tau) = -\frac{\vartheta_4^4(\tau)}{\vartheta_2^4(\tau)}, \tag{3.6}$$

whose values at the cusps are $h_2(0) = 0$, $h_2(1) = 1$, and $h_2(i\infty) = \infty$. Since its pole and zero are simple, h_2 qualifies as a $\Gamma(2)$ hauptfunktion. In fig. 4, we show the image of $\mathscr{F}_2$ under h_2. Any other function which maps $\mathscr{F}_2$ 1-1 onto $\hat{\mathbb{C}}$ could equally well serve as the hauptfunktion, e.g., the Picard λ-function, which is related to h_2 by $\lambda = 1/(1 - h_2) = \vartheta_2^4/\vartheta_3^4$.

(2) h_4. We only need to square the ϑ_i's to eliminate the phase if we are considering its transformations under $\Gamma(4)$, so the ϑ_i^2 are modular forms of weight 1 and level 4. From these we define the $\Gamma(4)$ hauptfunktion

$$h_4(\tau) = \frac{\vartheta_3^2 + \vartheta_4^2}{\vartheta_2^2}, \tag{3.7}$$

which maps the set of cusps of $\mathscr{F}_4$, $\{\frac{1}{2}, 0, 1, 2, -1, i\infty\}$, to $\{0, 1, -i, -1, i, \infty\} \in \hat{\mathbb{C}}$, respectively (see fig. 5).

(3) h_3 and h_5. The classical theta functions for a Lie group G are

$$\Theta_\lambda^G(\tau) = \sum_{\gamma \in M + \lambda} e^{i\pi\tau|\gamma|^2}, \tag{3.8}$$

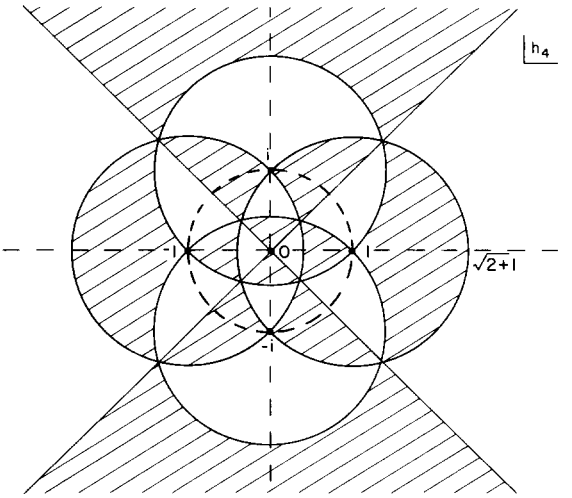

Fig. 5. The image of the map h_4: $\mathscr{F}_4 \to \hat{\mathbb{C}}$.

where M is the root lattice and λ is the highest weight of an irreducible representation. Since λ is only defined modulo the root lattice, choose it to lie within a unit cell of the same.

To construct the hauptfunktion for h_3, we make use of the fact that the Θ_λ functions for SU(3) are modular forms of level 3. λ labels the 1, 3 and $\bar{3}$ representations, but the Θ_λ functions for the latter two are equivalent

$$\Theta_1(\tau) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(6\tau)\,\vartheta\begin{bmatrix}0\\0\end{bmatrix}(2\tau) + \vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(6\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(2\tau),$$

$$\Theta_3(\tau) = \Theta_{\bar{3}}(\tau) = \vartheta\begin{bmatrix}\frac{1}{3}\\0\end{bmatrix}(6\tau)\,\vartheta\begin{bmatrix}0\\0\end{bmatrix}(2\tau) + \vartheta\begin{bmatrix}\frac{1}{6}\\0\end{bmatrix}(6\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(2\tau). \qquad (3.9)$$

(Our theta function conventions are given in appendix A.) From these we define

$$h_3(\tau) = \frac{\Theta_1(\tau)}{\Theta_3(\tau)}, \qquad (3.10)$$

which maps the cusps 0, 1, 2 and $i\infty$ to 1, $\bar{\rho}$, ρ and ∞.

Similarly, the classical Θ_λ functions for SU(5) are modular forms of level 5, and h_5 may be defined as a ratio of these, although later we will encounter a much easier way to define it.

$\Gamma(N)$ transformations leave $\mathscr{F}_N$ invariant, while elements of Γ not in $\Gamma(N)$ permute the modular triangles composing $\mathscr{F}_N$, inducing an automorphism which

maps cusps to cusps, i-points to i-points, etc. These automorphisms form a representation of $\Gamma_N = \Gamma/\Gamma(N)$, the modulary group of level N. For $N = 2$, 3, 4, and 5, the modulary groups are isomorphic to the rotation groups of the regular solids

$$\Gamma_2 \simeq S_3 \simeq \text{triangle group},$$

$$\Gamma_3 \simeq A_4 \simeq \text{tetrahedral group},$$

$$\Gamma_4 \simeq S_4 \simeq \text{octahedral group},$$

$$\Gamma_5 \simeq A_5 \simeq \text{icosahedral group}. \tag{3.11}$$

It is not obvious at first sight how, e.g., the rotation group of the octahedron acts on $\mathscr{F}_4$, but its action on the image of $\mathscr{F}_4$ under h_4 (fig. 5) is much more transparent. In fact, an inverse stereographic projection of the h_4 plane maps fig. 5 onto a spherical octahedron, with the cusps of $\mathscr{F}_4$ at the vertices, the i-points at the midpoints of the edges, and the ρ-points at the centers of the faces. Elements of Γ_4 simply rotate this octahedron, mapping vertices to vertices, etc. By mapping back, we can easily determine how the cusps transform under the modulary group. Furthermore, since functions are specified by the locations of their zeros and poles (i.e., by their divisors), we can see how h_4 and other level-4 modular functions transform under coset transformations by observing the motion of their zeros and poles. Similar constructions hold for the other values of $N \leqslant 5$.

4. Factorization on the modular group

We begin the process of classifying factorizable partition functions by considering the simplest case: complete holomorphic factorization on the modular group. Each term $f\bar{g}$ in eq. (1.1) is separately modular invariant, so that f and g are modular functions with multipliers, i.e., under a modular transformation they transform into themselves up to a τ-independent phase, $f(A\tau) = \chi_f(A) f(\tau) = \exp(i\phi_f(A)) f(\tau)$, and the phases must match, $\chi_f(A) = \chi_g(A)$, $\forall A \in \Gamma$. If the partition function is diagonal, i.e., of the form $\sum_{i=1}^{M} \bar{f}_i f_i$, the latter condition automatically holds. For brevity, we will hereafter refer to modular functions with multipliers as *projective modular functions*.

First, we consider modular functions without multipliers, $\chi_f(A) = 1$. f is then a rational function of j (2.10). Any term $j - b_j$ in the denominator, however, must vanish somewhere in the upper half plane, spoiling holomorphicity. Hence, f must be a polynomial in j, whose only poles lie at the cusp, which physically corresponds to the presence of a tachyon if we view the associated conformal field theory as a component of a compactification of the bosonic string. The conformal charge is $c = 24n$, where n is the order of the polynomial. In applications to string theory, we

are only interested in $c \leqslant 24$, so $n = 1$ and

$$f = j + a = \frac{1}{q} + (744 + a) + 196884q + \ldots . \tag{4.1}$$

Since j by itself has integer coefficients, a must also be an integer, and unitarity requires that $a \geqslant -744$. That f be a sum of Virasoro characters with positive coefficients (requirement R_5) puts an upper bound on a: e.g., consideration of the next term alone constrains $a \leqslant 196139$. We conclude that a partition function satisfying complete holomorphic factorization (without multipliers) must be composed of functions of the extremely limited form $f = j + a$, $a \in \mathbb{Z}$, $-744 \leqslant a \leqslant 196139$.

Such partition functions arise from compactifications of the bosonic string on the twenty-four 24-dimensional even self-dual Niemeier lattices, for which $f = \Theta_\Lambda / \eta^{24}$, where $\Theta_\Lambda = \sum_{\gamma \in \Lambda} e^{i\pi\tau|\gamma|^2}$ is the theta function of the lattice Λ and is a modular form of weight 12 [12]. The partition functions for these theories are given by

$$f = (j - 744) + 24(h + 1), \tag{4.2}$$

where h is the Coxeter number of the Niemeier lattice and can take on the values $h = 0, 2, \ldots, 10, 12, 13, 14, 16, 18, 22, 25, 30$ and 46 [16]. Partition functions of this form can also arise in asymmetric orbifolds based on these lattices. $\mathbb{Z}_2$-twists by the automorphism which takes lattice vectors to their negatives lead to partition functions given by

$$f = (j - 744) + 12h. \tag{4.3}$$

To find partition functions with $c < 24$, we must generalize to non-trivial multipliers. A multiplier system $\chi(A)$ is a multiplicative abelian character of the modular group, i.e., a representation of Γ in $\mathbb{E} = \{z \in \mathbb{C} \mid |z| = 1\}$, and is completely characterized by the multipliers of the generators S and T. Because of the relations $S^2 = (ST)^3 = 1$, we have $\chi(S) = \pm 1$ and $\chi(ST) = e^{2\pi i n/3}$, $n = 1$, 2, or 3, so that $\chi(T)$ is a sixth root of unity. This implies $\chi_f(A)^6 = 1$, $\forall A \in \Gamma$, so that f^6 is a modular function without multipliers, and f is an algebraic function of j, at worst a sixth-root. In general, $j + a$ has a simple pole at the cusp and a simple zero somewhere in the upper half plane, so that taking any root would introduce a branch point there, violating holomorphicity. But if $a = 0$ or -1728, $j + a$ vanishes at one of the orbifold points, where the order of the zero is higher by eq. (2.5). j has a triple zero at ρ, whence $\sqrt[3]{j}$ is holomorphic in $\mathscr{H}$. $j - 1728$ vanishes to second order at i so that $\sqrt{j - 1728}$ is likewise well behaved. Consequently, we can write

four holomorphic projective modular functions with $c < 24$

$$\sqrt[3]{j} = \frac{\tilde{G}_4}{\eta^8} = q^{-1/3}(1 + 248q + \ldots), \tag{4.4}$$

$$\sqrt{j - 1728} = \frac{\tilde{G}_6}{\eta^{12}} = q^{-1/2}(1 - 492q + \ldots), \tag{4.5}$$

$$\sqrt[3]{j^2} = \frac{\tilde{G}_4^2}{\eta^{16}} = q^{-2/3}(1 + 496q + \ldots), \tag{4.6}$$

$$\sqrt[6]{j^2(j - 1728)^3} = \frac{\tilde{G}_4\tilde{G}_6}{\eta^{20}} = q^{-5/6}(1 - 244q + \ldots). \tag{4.7}$$

All of these expressions have integer coefficients in their cusp expansions, but eqs. (4.5) and (4.7) are ruled out by requirement R_4 because some of the coefficients are negative. The other two satisfy all our requirements, and are partition functions for conformal field theories. Eq. (4.4) corresponds to compactification of eight dimensions on Γ_8, the root lattice of E_8, and eq. (4.6) corresponds to compactification of 16 dimensions on either Γ_8^2 or Γ_{16}, the root lattices of $E_8 \times E_8$ and $\mathrm{Spin}(32)/Z_2$ respectively, both of which lead to the same partition function [12].

5. Factorization on genus-zero subgroups

In this section, we turn our attention to partition functions which factorize on a genus-zero subgroup Γ', that is

$$Z = \sum_{i=1}^{M} f_i \bar{g}_i, \tag{5.1}$$

where f_i and g_i are modular functions, possibly with multiplier systems, of Γ', and the sum is required to gain full modular invariance. The form of such partition functions, after we have imposed the physical requirements $R_1 - R_5$, is strongly constrained, leading to restrictions on c and h and, in some cases, to a finite number of possibilities for Z.

Given a pair of projective modular functions of Γ', f and g, with the same multiplier systems, we define the modular invariant

$$\mathscr{I}(f, g) = \sum_{k=1}^{\mu} f(A_k\tau)\bar{g}(A_k\bar{\tau}), \tag{5.2}$$

where $\Gamma = \bigcup_{k=1}^{\mu} \Gamma' A_k$ is a coset decomposition of Γ' in Γ. Any Γ'-factorizable partition function is a linear combination of such modular invariants; indeed

$$Z = \frac{1}{\mu} \sum_{i=1}^{M} \mathscr{Y}(f_i, g_i), \tag{5.3}$$

so we turn to the possible forms for $\mathscr{Y}$.

First, we consider the case without multiplier systems. Any modular function of Γ' can be written as a rational function of the hauptfunktion h

$$f \propto \frac{\prod_{j=1}^{n}(h - a_j)}{\prod_{k=1}^{m}(h - b_k)}. \tag{5.4}$$

Since h takes every complex value exactly once on $\mathscr{F}'$, we can uniquely write this

$$f \propto \frac{\prod_{j=1}^{n}(h - h(\tau_j'))}{\prod_{k=1}^{m}(h - h(\tau_k))}, \tag{5.5}$$

where $\tau_k, \tau_j' \in \mathscr{F}'$. Holomorphicity in the upper half plane requires that all the poles lie at the cusps, i.e., $\tau_k \in \mathbb{Q} \cup \{i\infty\}$.

For the moment, we restrict our attention to modular functions of the genus-zero principal congruence subgroups, $\Gamma(N)$, $N \leqslant 5$. These functions are invariant under $T^N \in \Gamma(N)$: $\tau \to \tau + N$, so they have cusp expansions $f = \sum_l f_l e^{2\pi i \tau l/N} = \sum_l f_l q^{l/N}$. A function with a simple pole at $i\infty$ has leading term $q^{-1/N}$, which corresponds to $c = c_N = 24/N$. Since each cusp is mapped to $i\infty$ under some coset transformation, the leading behavior of the modular invariant $\mathscr{Y}$ constructed from f is determined by the pole of largest order, $\tilde{n}$, giving $c = \tilde{n} c_N$. Unlike the poles, the positions of the zeros of f remain unconstrained, which leaves great freedom for constructing modular invariants, and from these, partition functions, out of modular functions of level N. (Eventually, we must impose $R_2 - R_5$, which will lead to some restrictions on these.) The conformal charge for these partition functions, however, must necessarily be a multiple of $c_N = 12, 8, 6, \frac{24}{5}$ for $N = 2, 3, 4, 5$. To obtain any other value of c, particularly a lower one, we must allow the more general possibility of projective modular functions of level N.

For the full modular group, the relations satisfied by the generators imply that all multipliers must be sixth roots of unity. This means that the number of possible multiplier systems for Γ is finite. Such an idyllic situation does not hold in general for modular subgroups. The multiplier systems for Γ' form a group, the group of multiplicative abelian characters, which is isomorphic to Γ'/Γ_C', where Γ_C' is the commutator subgroup of Γ'. (The commutator subgroup is generated by group elements of the form $ABA^{-1}B^{-1}$ for $A, B \in \Gamma'$).

For the index two subgroup Γ^2 described in sect. 3, $\Gamma^2/\Gamma_C^2 \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, so that any multiplier system cubed is the unit multiplier system, hence any multiplier is a cube root of unity. Similarly, $\Gamma^3/\Gamma_C^3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, so that all multipliers are ± 1. The fundamental regions for Γ^2 and Γ^3 are the branched coverings of $\mathscr{F}$ on which the functions $\sqrt{j-1728}$ and $\sqrt[3]{j}$, respectively, are single-valued, so that these are hauptfunktions for these genus-zero subgroups. Using these facts, it is not too hard to see that holomorphic projective modular functions of Γ^2 and Γ^3 are also holomorphic projective modular functions of the full modular group, so that partition functions that factorize on these subgroups also factorize on Γ, and we obtain nothing new.

For the other normal genus-zero subgroups, $\Gamma(N)$, $2 \leqslant N \leqslant 5$, the commutator subgroup has infinite index, whence the number of multiplier systems is infinite. The reason is that the $\Gamma(N)$ are free groups, i.e., their generators satisfy no relations and therefore can have arbitrary multipliers. This arbitrariness of the multipliers could spell disaster, since we can write a projective modular function of level $N \leqslant 5$ as an algebraic function of h_N if and only if the phases of its multipliers (divided by 2π) are rational. This is, in fact, the case, however, for projective modular functions appearing in rational conformal field theories. Anderson and Moore [6] have recently given a proof that rational conformal field theories have rational values of c and h. A simple consequence of this is that the phases of the multiplier system of a projective modular function of level N must be rational, as we now show. The subgroup $\Gamma(N)$ for $N \leqslant 5$ is generated by the boundary substitutions of $\mathscr{F}_N$ across the cusps, i.e., by those elements of $\Gamma(N)$ which map the two line segments on the boundary of $\mathscr{F}_N$ which connect to the same cusp into one another. (Only $\sigma_\infty - 1$ of these generators are independent.) Now suppose that the phase of f under the generator T^N is $2\pi\beta_1$, i.e., $f(\tau+N) = e^{2\pi i\beta_1} f(\tau)$. Then $q^{-\beta_1/N} f(\tau)$ is invariant under T^N and $f(\tau) = q^{\beta_1/N} \sum_{l \in \mathbb{Z}} f_l q^{l/N}$. Let $2\pi\beta_\nu$ be the phase of f under the generator associated with the boundary substitutions across the cusp c_ν. If $A_\nu \in \Gamma$ maps that cusp to $i\infty$, then the coset transform of f by A_ν has cusp expansion $f(A_\nu\tau) = \hat{f}(\tau) = q^{\beta_\nu/N} \sum_{l \in \mathbb{Z}} \hat{f}_l q^{l/N}$. Now the conformal charge for $\mathscr{V}(f,f)$ is determined by the coset transform whose leading term has the most negative exponent, say $f(\tilde{A}\tau)$, and the conformal weights by the differences of the exponents of the different transforms

$$c = \frac{24\tilde{\beta}}{N} \bmod \frac{1}{N}, \qquad h_\nu = \frac{\beta_\nu - \tilde{\beta}}{N} \bmod \frac{1}{N}. \tag{5.6}$$

Since c and h are rational, the phases β_ν of the multipliers of the generators, and therefore of all elements of $\Gamma(N)$, must also be rational. Thus, every projective modular function f of level $N \leqslant 5$ which appears in the partition function of a rational conformal field theory (5.1) is a root of some modular function without multipliers, which we will call $F(\tau)$.

We will now argue that we lose no generality by only considering what we will call *restricted* modular functions, those whose poles and zeros all lie at the cusps. If F has a zero in the upper half plane, then unless the order of the zero is a multiple of r, the projective function $f = \sqrt[r]{F}$ will have a branch point there, in violation of R_1, the holomorphicity requirement on f. Because $n_i = 2$ and $n_\rho = 3$ for the principal congruence subgroups, the points i and ρ are no longer orbifold points in $\mathscr{F}_N$, and are to be treated as any other point in the upper half plane. Let the zeros of F which are located at points τ_i in the upper half plane (i.e., excluding the cusps) have orders n_i such that $r | n_i$, $\forall i$. Let h be the hauptfunktion for a genus zero subgroup, and if necessary subtract a constant from h so that its zero lies at one of the cusps. Then we can write F in the form

$$F(\tau) = \prod_i (h - h(\tau_i))^{n_i} F'(\tau), \tag{5.7}$$

where F' is a function with all its poles and zeros at the cusps. Since $n_i/r \in \mathbb{Z}$, the projective function

$$f(\tau) = \prod_i (h - h(\tau_i))^{n_i/r} \sqrt[r]{F'} \tag{5.8}$$

can be expanded into a sum of restricted projective modular functions, and thus the modular invariant made from f is a linear combination of modular invariants made from restricted projective functions. Hence, we need only consider restricted modular functions F.

If $c_1, \ldots, c_{\sigma_\infty} \in \mathbb{Q} \cup \{i\infty\}$ are the cusps of $\mathscr{F}_N$, then F has divisor

$$(F) = c_1^{n_1} \ldots c_{\sigma_\infty}^{n_{\sigma_\infty}}, \tag{5.9}$$

where $n_\nu \in \mathbb{Z}$ is the order of the zero (or pole, if it be negative) of F at c_ν, and $\sum_{\nu=1}^{\sigma_\infty} n_\nu = 0$. We will also write the divisor in the abbreviated form $(n_{c_1}, \ldots, n_{c_{\sigma_\infty}})$, using the cusp itself as the subscript. The divisor completely determines the function up to normalization. Now consider the projective modular function $f = \sqrt[r]{F}$. The cusp expansion of a coset transform A_ν of f which takes the νth cusp to $i\infty$ will have the form

$$f(A_\nu \tau) = \sqrt[r]{F(A_\nu \tau)} = q^{-n_\nu/Nr}\left(a_{0,\nu} + a_{1,\nu} q^{1/N} + \ldots\right). \tag{5.10}$$

The conformal charge for $\mathscr{A}(f, f)$ is $c = 24\tilde{n}/Nr$, where $\tilde{n}$ is the order of the maximal pole of F, $\tilde{n} = |\min n_\nu|$. The conformal weights are given by $h_\nu = \Delta n_\nu / Nr = \frac{1}{24} c(\Delta n_\nu/\tilde{n})$, where $\Delta n_\nu = n_\nu + \tilde{n}$. This already allows us to derive a constraint on the conformal weights. Since $\sum_{\nu=1}^{\sigma_\infty} (\Delta n_\nu/\tilde{n}) = \sigma_\infty$, each term in the sum obeys $(\Delta n_\nu/\tilde{n}) \leqslant \sigma_\infty$, so that the conformal weights coming from the leading terms of the

coset transforms satisfy $h_\nu \leqslant \frac{1}{24}c\sigma_\infty = \frac{1}{24}(\mu c/N)$, where μ is given in table 1. Subleading terms contribute conformal weights $h_\nu + l/N$, $l \in \mathbb{Z}$, so for $c < 24/\mu$, we have the restriction

$$0 \leqslant h \bmod \frac{1}{N} \leqslant \frac{\mu c}{24N}. \tag{5.11}$$

We will be able to obtain much stronger restrictions for c and h when we consider specific subgroups in sects. 6 and 7.

If we wish to construct non-diagonal modular invariants $\mathscr{Y}(f, g)$, the multiplier systems for these projective functions must be the same. What constraint does this place on the modular functions $F = f^r$ and $G = g^r$? For a function F with a zero of order n_ν at some cusp, the phase under the generator of $\Gamma(N)$ corresponding to the boundary substitution across that cusp is $2\pi n_\nu$, and so the phase of f is $2\pi n_\nu/r$. In order that the multipliers of f and g for the generators be equal, the phases must be the same modulo 2π, i.e., $n_\nu(F) = n_\nu(G) \bmod r$ at each of the cusps. That is, F and G must have the same divisors mod r. This restriction will result in many fewer possibilities for non-diagonal modular invariants than for diagonal ones.

We have learned that we should only take roots of restricted modular functions F, but since *a priori* there are no constraints on the phases of the multiplier system other than that they be rational, how do we know *which* roots of F we may take? Here we appeal to requirement R_3, that the coefficients of the cusp expansion of Z must be integral. It is not difficult to show that the coefficients of the cusp expansion of any restricted modular function F belong to $\mathbb{Z}(\zeta_N)$, e.g., by writing it in terms of the cusp functions, which we will define in sect. 7. ($\mathbb{Z}(\zeta_N)$ is the ring of cyclotomic integers, i.e., complex numbers of the form $l_1\zeta_N + l_2\zeta_N^2 + \cdots + l_N$, where $\zeta_N = e^{2\pi i/N}$, and $l_i \in \mathbb{Z}$.) We will argue that for R_3 to hold, the coefficients of the cusp expansions of the projective functions $f = \sqrt[r]{F}$, from which we build the modular invariants $\mathscr{Y}$, must also lie in $\mathbb{Z}(\zeta_N)$, when the leading term is normalized to unity. This will provide strong constraints on the allowed projective functions, and on the values of c and h in the resulting partition functions. In fact, for $c < c_N$, there will be a small number of such projective functions, which we will show how to find, and any $\Gamma(N)$ factorizable partition function with $c < c_N$ must be a linear combination of the modular invariants formed out of these allowed projective functions.

6. Factorization on $\Gamma(2)$

In this section and sect. 7, we will make concrete the procedure presented in sect. 5 by applying it to the genus-zero principal congruence subgroups.

We specify a restricted modular function F of level 2 by its divisor $(n_0, n_1, n_{i\infty})$. The sum over cosets results in all permutations of these cusps, so, without loss of

generality, we may take $\tilde{n} = -n_{i\infty} = n_0 + n_1$ to be the pole of largest order. The explicit form of F, up to normalization, is

$$F \propto h_2^{n_0}(h_2 - 1)^{n_1} = \left(-\frac{\vartheta_4^4}{\vartheta_2^4}\right)^{n_0}\left(-\frac{\vartheta_3^4}{\vartheta_2^4}\right)^{n_1} \propto \left(\frac{\vartheta_4^{2n_0+n_1}\vartheta_3^{n_0+2n_1}}{\eta^{3n_0+3n_1}}\right)^4$$

$$= \left(\frac{\vartheta_4^{m_0}\vartheta_3^{m_1}}{\eta^{m_0+m_1}}\right)^{4m} = \left(\eta_0^{m_0}\eta_1^{m_1}\right)^{8m}, \tag{6.1}$$

where we define

$$\eta_0(\tau) \equiv \sqrt{\frac{\vartheta_4}{\eta}} = q^{-1/48}\prod_{m=0}^{\infty}\left(1 - q^{m+1/2}\right),$$

$$\eta_1(\tau) \equiv \sqrt{\frac{\vartheta_3}{\eta}} = q^{-1/48}\prod_{m=0}^{\infty}\left(1 + q^{m+1/2}\right), \tag{6.2}$$

and where m is the greatest common divisor $(2n_0 + n_1, n_0 + 2n_1)$, and $m_0 = (2n_0 + n_1)/m$ and $m_1 = (n_0 + 2n_1)/m$.

We now come to the crucial point. The projective modular functions η_0 and η_1 have all integer coefficients in their cusp expansions, as is manifest from their product representations (6.2). The coefficients in the cusp expansions of *roots* of these functions, however, will be fractions, and in fact the denominators of these rational coefficients will grow with the power of q, as can be seen from the binomial expansion of the product representation. Because of this, the *finite* sum over cosets (5.2) and over modular invariants (5.3) cannot restore the integrality of all the coefficients of the partition function, violating R_3. Hence, $f = \sqrt[r]{F}$ must be a product of integral powers of η_0 and η_1 (i.e., must itself have only integer coefficients in its cusp expansion), which occurs if and only if r divides $8m$.

Let $l = 8m/r \in \mathbb{Z}$. Since the coset transformations simply permute the three Jacobi theta functions in all possible ways, f gives rise to the modular invariant

$$\mathscr{Y}(f,f) = \frac{1}{\mathscr{N}}\left(\left|\frac{\vartheta_4^{m_0}\vartheta_3^{m_1}}{\eta^{m_0+m_1}}\right|^l + \left|\frac{\vartheta_4^{m_0}\vartheta_2^{m_1}}{\eta^{m_0+m_1}}\right|^l + \left|\frac{\vartheta_3^{m_0}\vartheta_4^{m_1}}{\eta^{m_0+m_1}}\right|^l\right.$$

$$\left. + \left|\frac{\vartheta_3^{m_0}\vartheta_2^{m_1}}{\eta^{m_0+m_1}}\right|^l + \left|\frac{\vartheta_2^{m_0}\vartheta_4^{m_1}}{\eta^{m_0+m_1}}\right|^l + \left|\frac{\vartheta_2^{m_0}\vartheta_3^{m_1}}{\eta^{m_0+m_1}}\right|^l\right), \tag{6.3}$$

where we divide by a normalization factor $\mathscr{N}$ so that there is only one identity operator, and then must recheck that the coefficients of $\mathscr{Y}$ are still integers. The

conformal charge of this modular invariant is

$$c = \tfrac{1}{2} l(m_0 + m_1) \in \tfrac{1}{2}\mathbb{Z}, \tag{6.4}$$

and its conformal weights are

$$h = 0, \tfrac{1}{16} l m_0, \tfrac{1}{16} l m_1 \bmod \tfrac{1}{2} \in \tfrac{1}{16}\mathbb{Z}. \tag{6.5}$$

For $c < 4$, eq. (6.5), together with eq. (5.11), tells us that

$$h = 0, \tfrac{1}{16}, \ldots, \tfrac{1}{8} c \bmod \tfrac{1}{2}, \tag{6.6}$$

whereas for $c \geqslant 4$, all multiples of $\frac{1}{16}$ are allowed.

Recasting the problem slightly, we consider the divisor of the projective modular function f itself, $(p_0, p_1, p_{i\infty}) = (n_0/r, n_1/r, n_{i\infty}/r)$. This involves a slight abuse of notation since f is only well defined on a multisheeted covering surface of $\mathscr{F}_2$ with the values of f at equivalent points on different sheets related by a phase determined by the fractional part of its divisor. Since

$$f = \eta_0^{8(2p_0+p_1)} \eta_1^{8(p_0+2p_1)}, \tag{6.7}$$

its coefficients will be integer-valued if and only if the powers of η_0 and η_1 in eq. (6.7) are integers. This implies that $p_\nu \in \frac{1}{24}\mathbb{Z}$ and the difference of any two p_ν's $\in \frac{1}{8}\mathbb{Z}$, from which eqs. (6.4) and (6.5) follow, since $c = 24\tilde{p}/N$, $\tilde{p} = \tilde{n}/r$, and $h_\nu = \Delta p_\nu / N$, $\Delta p_\nu = p_\nu + \tilde{p}$. For a given value of c, there exist $[c] + 1$ distinct (up to coset transformations) level-2 projective modular functions with integer coefficients, whose divisors we list in table 2 for $c \leqslant 2$.

We will now see that the modular invariants constructed from these f_i correspond to the partition functions of known conformal field theories. Defining $\mathscr{Y}_i = \mathscr{Y}(f_i, f_i)$,

TABLE 2
Divisors of level 2 projective functions with integer coefficients for $c \leqslant 2$

c	$f = (p_0, p_1, p_{i\infty})$	$h \bmod \frac{1}{2}$
$\frac{1}{2}$	$f_1 = (-\frac{1}{24}, -\frac{1}{24}, \frac{1}{12})$	$0, \frac{1}{16}$.
1	$f_2 = (-\frac{1}{12}, -\frac{1}{12}, \frac{1}{6})$	$0, \frac{1}{8}$.
1	$f_3 = (-\frac{1}{12}, \frac{1}{24}, \frac{1}{24})$	$0, \frac{1}{16}$.
$\frac{3}{2}$	$(-\frac{1}{8}, -\frac{1}{8}, \frac{1}{4})$	$0, \frac{3}{16}$.
$\frac{3}{2}$	$(-\frac{1}{8}, 0, \frac{1}{8})$	$0, \frac{1}{16}, \frac{1}{8}$.
2	$(-\frac{1}{6}, -\frac{1}{6}, \frac{1}{3})$	$0, \frac{1}{4}$.
2	$(-\frac{1}{6}, -\frac{1}{24}, \frac{5}{24})$	$0, \frac{1}{16}, \frac{3}{16}$.
2	$(-\frac{1}{6}, \frac{1}{12}, \frac{1}{12})$	$0, \frac{1}{8}$.

normalized so that the coefficient of the leading term is unity, we observe that

$$\mathscr{Y}_1 = \frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right| + \left|\frac{\vartheta_3}{\eta}\right| + \left|\frac{\vartheta_4}{\eta}\right|\right) \tag{6.8}$$

is the partition function for one free Majorana fermion, the Ising model, which has $c = \frac{1}{2}$. The partition function for two Majorana fermions with correlated spin structures, or equivalently, for a boson whose target space is a circle of radius $R = 1$ is given by

$$\mathscr{Y}_2 = \frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right|^2 + \left|\frac{\vartheta_3}{\eta}\right|^2 + \left|\frac{\vartheta_4}{\eta}\right|^2\right). \tag{6.9}$$

The other $\Gamma(2)$-factorizable modular invariant at $c = 1$

$$\mathscr{Y}_3 = \left(\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right| + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right| + \left|\frac{\vartheta_4\vartheta_2}{\eta^2}\right|\right), \tag{6.10}$$

by itself has negative coefficients, but the linear combination

$$\tfrac{1}{2}(\mathscr{Y}_2 + \mathscr{Y}_3) = \mathscr{Y}_1^2 = \frac{1}{4}\left(\left|\frac{\vartheta_2}{\eta}\right| + \left|\frac{\vartheta_3}{\eta}\right| + \left|\frac{\vartheta_4}{\eta}\right|\right)^2, \tag{6.11}$$

has all positive integer coefficients, and is the partition function for two uncorrelated Majorana fermions, or, equivalently, for a boson on a $\mathbb{Z}_2$ orbifold of the circle with $R = 1$. In fact, it can be shown that eqs. (6.9) and (6.11) are the only linear combinations of the two $\Gamma(2)$-factorizable modular invariants at $c = 1$ that satisfy the requirements $\mathrm{R}_1 - \mathrm{R}_5$ which we imposed on allowable partition functions.

The conformal charges (6.4) and weights (6.6) are exactly those that may appear in theories that are constructed out of free Majorana fermions. Certainly, all such theories factorize on $\Gamma(2)$, and it is tempting to conjecture the converse, that all $\Gamma(2)$-factorizable theories can be so constructed, though we do not yet have a proof of this.

We briefly consider non-diagonal modular invariants, $\mathscr{Y}(f, g)$. The condition on the projective functions from which we may construct non-diagonal invariants, viz., $p_\nu(f) = p_\nu(g)$ mod 1 for each cusp c_ν, is first satisfied at $c = 4$, with $(f) = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$ and $(g) = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$. This yields

$$\mathscr{Y}(f, g) = \frac{1}{2\bar{\eta}\eta}\left(\bar{\vartheta}_2^4\vartheta_3^4 - \bar{\vartheta}_2^4\vartheta_4^4 + \bar{\vartheta}_3^4\vartheta_4^4 + \bar{\vartheta}_3^4\vartheta_2^4 - \bar{\vartheta}_4^4\vartheta_2^4 + \bar{\vartheta}_4^4\vartheta_3^4\right), \tag{6.12}$$

which by Jacobi's identity is the same as the diagonal invariant $\mathscr{Y}(f, f) = \mathscr{Y}(g, g)$, the partition function for eight correlated Majorana fermions. In general, the number of non-diagonal modular invariants is far fewer than diagonal ones.

7. Factorization on $\Gamma(4)$, $\Gamma(3)$ and $\Gamma(5)$

A restricted projective modular function of level 4 whose divisor is $(p_0, p_2; p_1, p_{-1}; p_{1/2}, p_{i\infty})$, where the orders $p_\nu = n_\nu/r$ are now fractions, can be explicitly written in terms of the $\Gamma(4)$ hauptfunktion

$$f \propto (h_4 - 1)^{p_0}(h_4 + 1)^{p_2}(h_4 + i)^{p_1}(h_4 - i)^{p_{-1}}(h_4)^{p_{1/2}}. \tag{7.1}$$

Even with the explicit form of h_4 (3.7), however, this expression is not very useful for determining whether the cusp expansion of f has coefficients in $\mathbb{Z}(i)$ or not. For this purpose, we develop a product representation of f.

We define a set of *cusp functions of level N*, $\eta_c^{(N)}(\tau)$, one for each $\Gamma(N)$-inequivalent cusp c, which are projective modular functions of level N. The function $\eta_c^{(N)}$ has the property that each factor in its product expansion vanishes at a set of cusps $\Gamma(N)$-equivalent to c, and each factor appears exactly once. $\eta_c^{(N)}$ vanishes at $\tau = c$ and has all its other zeros and poles at other cusps. We label them η functions because they are essentially the generalized Dedekind eta functions $\eta_{(g,h)}^{(N)}$ [11], except for $N = 2$, where they differ by a power of two (see appendix A for definitions and properties). To make the equations less cumbersome, we will suppress the superscript (N) in the following, since the context should make clear which level we mean. The cusp functions of level 4, for example, are

$$\eta_0 = \eta_{(1,0)} = q^{-1/96} \prod_{n=0}^{\infty} \left(1 - q^{n/2+1/4}\right),$$

$$\eta_2 = \eta_{(1,2)} = q^{-1/96} \prod_{n=0}^{\infty} \left(1 + q^{n/2+1/4}\right),$$

$$\eta_1 = \eta_{(1,3)} = q^{-1/96} \prod_{n=0}^{\infty} \left(1 + (-)^n i q^{n/2+1/4}\right),$$

$$\eta_{-1} = \eta_{(1,1)} = q^{-1/96} \prod_{n=0}^{\infty} \left(1 - (-)^n i q^{n/2+1/4}\right),$$

$$\eta_{1/2} = \eta_{(2,1)} = q^{-1/24} \prod_{n=0}^{\infty} \left(1 + q^{2n+1}\right),$$

$$\eta_{i\infty} = \eta_{(0,1)} = \sqrt{2}\, q^{1/12} \prod_{n=1}^{\infty} \left(1 + q^{2n}\right). \tag{7.2}$$

The η_c's transform into each other up to a phase under modular transformations: $\eta_{Ac}(A\tau) \propto \eta_c(\tau)$. To determine the order of η_c at some cusp c', we find $A \in \Gamma$ which takes c' to $i\infty$ and then check the order of η_{Ac} at $i\infty$, i.e., the leading power of q in units of $1/N$. Thus, e.g., $\eta_{i\infty}$ has divisor $(-\frac{1}{24}, -\frac{1}{24}; -\frac{1}{24}, -\frac{1}{24}; -\frac{1}{6}, \frac{1}{3})$.

In sect. 3, we constructed h_N, the hauptfunktion for $\Gamma(N)$, by taking a ratio of modular forms. We just mention here that h_N can also be expressed as a product of cusp functions having a simple zero at one cusp and a simple pole at another. In fact, *any* restricted projective modular function can be written as a product of powers of the η_c's. For example, the function (7.1) can be expressed

$$f \propto \eta_0^{3p_0+p_2}\eta_2^{3p_2+p_0}\eta_1^{3p_1+p_{-1}}\eta_{-1}^{3p_{-1}+p_1}\eta_{1/2}^{3p_{1/2}+p_{i\infty}}\eta_{i\infty}^{3p_{i\infty}+p_{1/2}}, \tag{7.3}$$

as can be seen by comparing their divisors. This expression is not unique: each of the exponents can be shifted by an equal amount because the product of all the η_c's is a constant, $\eta_0\eta_2\eta_1\eta_{-1}\eta_{1/2}\eta_{i\infty} = \sqrt{2}$. (To see this, note that the divisor of the product of the cusp functions is $(0,0;0,0;0,0)$, hence it must be a constant, and then evaluate the constant at $\tau = i\infty$.) The next question is which of these restricted projective modular functions can give rise to partition functions satisfying R_3.

The coefficients in the cusp expansions of the $\eta_c^{(N)}$'s lie in $\mathbb{Z}(\zeta_N)$, as is evident from their definitions. Projective modular functions f which are products of *integral powers* of cusp functions therefore also have cyclotomic integral coefficients. Of course, the partition function itself must have coefficients in $\mathbb{Z}$, not $\mathbb{Z}(\zeta_N)$, but the sum over cosets (5.2) usually results in a modular invariant $\mathscr{Y}(f, f)$ with $\mathbb{Z}$-coefficients only. As before, however, roots of cusp functions have coefficients in $\mathbb{Q}(\zeta_N)$, with denominators that grow with q. The same holds for functions which are products of non-integral powers of the $\eta_c^{(N)}$'s, so that modular invariants formed from these (which only involve finite sums) cannot have integer coefficients for all powers of q. Hence, for f to lead to a physically acceptable partition function, it must have coefficients which are cyclotomic integers.

In order for the coefficients of the cusp expansion of eq. (7.3) to be cyclotomic integers, the difference between any two exponents must be an integer. (We can then use eq. (A.13) to make all the exponents themselves integers.) We will refer to this as the integrality condition. Clearly, if f satisfies this requirement, so will its coset transforms, which are obtained from eq. (7.3) through certain permutations of the η_c's. The idea is then that the modular invariant (5.2) obtained by summing over coset transforms will have coefficients that are strictly integers (not just cyclotomic integers), although we must check this.

The integrality condition leads to the following necessary (but not sufficient) conditions on the divisor of f: $p_\nu \in \frac{1}{24}\mathbb{Z}$ and $p_\nu - p_{\nu'} \in \frac{1}{8}\mathbb{Z}$, from which it follows that for $\Gamma(4)$-factorizable partition functions

$$c \in \tfrac{1}{4}\mathbb{Z}, \qquad h \in \tfrac{1}{32}\mathbb{Z}. \tag{7.4}$$

TABLE 3
Divisors of level 4 projective functions with $\mathbb{Z}(i)$ coefficients for $c \leqslant \frac{5}{4}$

c	$f=(p_0,\ p_2;\ p_1,\ p_{-1};\ p_{1/2},\ p_{i\infty})$	$h \mod \frac{1}{4}$
$\frac{1}{2}$	$f_1=(-\frac{1}{12},\ -\frac{1}{12};\ -\frac{1}{12},\ -\frac{1}{12};\ \frac{1}{6},\ \frac{1}{6})$	$0,\ \frac{1}{16}\cdot$
$\frac{3}{4}$	$f_4=(-\frac{1}{8},\ -\frac{1}{8};\ \frac{1}{8},\ -\frac{1}{8};\ 0,\ \frac{1}{2})$	$0,\ \frac{1}{32},\ \frac{5}{32}\cdot$
1	$f_2=(-\frac{1}{6},\ -\frac{1}{6};\ -\frac{1}{6},\ -\frac{1}{6};\ \frac{1}{3},\ \frac{1}{3})$	$0,\ \frac{1}{8}\cdot$
1	$f_3=(-\frac{1}{6},\ -\frac{1}{6};\ \frac{1}{12},\ \frac{1}{12};\ \frac{1}{12},\ \frac{1}{12})$	$0,\ \frac{1}{16}\cdot$
1	$f_5=(-\frac{1}{6},\ -\frac{1}{6};\ -\frac{1}{6},\ -\frac{1}{6};\ -\frac{1}{6},\ \frac{5}{6})$	$0\cdot$
1	$f_6=(-\frac{1}{6},\ \frac{1}{3};\ -\frac{1}{24},\ -\frac{1}{24};\ -\frac{1}{24},\ -\frac{1}{24})$	$0,\ \frac{1}{32},\ \frac{1}{8}\cdot$
$\frac{5}{4}$	$(-\frac{5}{24},\ -\frac{5}{24};\ -\frac{5}{24},\ -\frac{5}{24};\ \frac{1}{6},\ \frac{2}{3})$	$0,\ \frac{3}{32},\ \frac{7}{32}\cdot$
$\frac{5}{4}$	$(-\frac{5}{24},\ -\frac{5}{24};\ -\frac{1}{12},\ \frac{5}{12};\ \frac{1}{24},\ \frac{1}{24})$	$0,\ \frac{1}{32},\ \frac{1}{16},\ \frac{5}{32}\cdot$
$\frac{5}{4}$	$(-\frac{5}{24},\ \frac{7}{24};\ -\frac{5}{24},\ \frac{7}{24};\ -\frac{1}{12},\ -\frac{1}{12})$	$0,\ \frac{1}{32},\ \frac{1}{8}\cdot$

It is not difficult to enumerate the divisors of all functions (up to coset transformations) which satisfy the integrality condition, and we list the results for $c \leqslant \frac{5}{4}$ in table 3.

Projective functions of level 4 whose divisors satisfy $p_0 = p_2 = 2\hat{p}_0$, $p_1 = p_{-1} = 2\hat{p}_1$ and $p_{1/2} = p_{i\infty} = 2\hat{p}_{i\infty}$ are actually projective functions of level 2, since these pairs of cusps are identified under $\Gamma(2)$, and

$$f \propto (\eta_0\eta_2)^{8\hat{p}_0}(\eta_1\eta_{-1})^{8\hat{p}_1}(\eta_{1/2}\eta_{i\infty})^{8\hat{p}_{i\infty}} = \hat{\eta}_0^{8\hat{p}_0}\hat{\eta}_1^{8\hat{p}_1}\hat{\eta}_{i\infty}^{8\hat{p}_{i\infty}}, \tag{7.5}$$

where the $\hat{\eta}_c$'s denote cusp functions of level 2, and in which the last expression is simply a more symmetric way of writing eq. (6.7). Hence, the projective functions f_1, f_2 and f_3 in table 3 are the same as those of the same name in table 2.

We digress briefly to describe some of the theories to which these modular invariants correspond. We have already met $\mathscr{Y}_1$ (6.8), the partition function of the Ising model. $\mathscr{Y}_4$, though satisfying the conditions we have so far imposed, certainly violates R_4 since the Virasoro characters are not unitary at $c = \frac{3}{4}$ [2]. The known conformal field theories at $c = 1$ consist of two one-parameter families of theories and three isolated theories [9,17]. A single free scalar field ϕ whose target space is a circle of radius R (so that $\phi \approx \phi + 2\pi R$) has the partition function

$$Z(R) = \frac{1}{\eta\bar{\eta}} \sum_{m,n\in\mathbb{Z}} q^{(m/2R+nR)^2/2}\bar{q}^{(m/2R-nR)^2/2}. \tag{7.6}$$

Since this line of theories manifestly satisfies the duality relation $Z(R) = Z(1/2R)$,

we may choose $R \geqslant 1/\sqrt{2}$ to label a set of distinct theories. For convenience, define

$$Z_k \equiv Z\left(\sqrt{k/2}\right) = \frac{1}{\eta\bar{\eta}} \sum_{m,n\in\mathbb{Z}} q^{(m+nk)^2/4k} \bar{q}^{(m-nk)^2/4k}$$

$$= \frac{1}{\eta\bar{\eta}} \sum_{l=0}^{2k-1} \sum_{s,t\in\mathbb{Z}} q^{(2ks+l)^2/4k} \bar{q}^{(2kt+l)^2/4k} = \frac{1}{\eta\bar{\eta}} \sum_{l=0}^{2k-1} |\Theta_{l,k}(\tau)|^2 , \qquad (7.7)$$

where

$$\Theta_{l,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{k(n+l/2k)^2} \qquad (7.8)$$

are the classical theta functions of level k for SU(2) [15]. The other line of theories consists of orbifold models obtained by modding out the target circle by the $\mathbb{Z}_2$ symmetry, $\phi \to -\phi$, which have the partition function

$$Z^{\text{orb}}(R) = \tfrac{1}{2}\left(Z(R) + 2Z_4 - Z_1\right) = \tfrac{1}{2}\left(Z(R) + \mathscr{Y}_3\right). \qquad (7.9)$$

As we saw in sect. 6, f_2 and f_3 give rise to the partition functions $Z_2 = \mathscr{Y}_2$ and $Z_2^{\text{orb}} = \frac{1}{2}(\mathscr{Y}_2 + \mathscr{Y}_3)$. From f_5, we obtain

$$Z_1 = \mathscr{Y}_5 = \left|\frac{\Theta_{0,1}}{\eta}\right|^2 + \left|\frac{\Theta_{1,1}}{\eta}\right|^2 , \qquad (7.10)$$

which in fact is the partition function for the level one WZW theory with gauge group SU(2). From eq. (7.9), we see that $Z_1^{\text{orb}} = Z_4 = \frac{1}{2}(\mathscr{Y}_5 + \mathscr{Y}_3)$ and $Z_4^{\text{orb}} = \frac{1}{4}(\mathscr{Y}_5 + 3\mathscr{Y}_3)$ also factorize on $\Gamma(4)$. Finally, the modular invariant arising from f_6,

$$\mathscr{Y}_6 = |\eta_0|^2 + |\eta_2|^2 + |\eta_1|^2 + |\eta_{-1}|^2 + |\eta_{1/2}|^2 + |\eta_{i\infty}|^2 , \qquad (7.11)$$

has weights 0, $\frac{1}{32}$, $\frac{1}{8}$ mod $\frac{1}{2}$, and so would seem to correspond to Z_8, which has the same weights. The multiplicities are wrong, however, and moreover $\mathscr{Y}_6$ violates R_5, so we need to take a linear combination of modular invariants, $Z_8 = \frac{1}{2}(\mathscr{Y}_6 + \mathscr{Y}_2)$. Lastly, $Z_8^{\text{orb}} = \frac{1}{4}(\mathscr{Y}_6 + \mathscr{Y}_2 + 2\mathscr{Y}_3)$ is also $\Gamma(4)$-factorizable. Because of the constraints on the weights (7.4), it is clear that these are all of the known $c = 1$ conformal field theories that factorize on $\Gamma(4)$. (None of the three isolated theories [17], which are obtained by modding out the theory corresponding to Z_1 by finite subgroups of SU(2), factorize on $\Gamma(4)$.)

We return to the other genus-zero principal congruence subgroups. A projective modular function of level 3 with divisor $(p_0, p_1, p_{-1}, p_{i\infty})$ can be written in terms

TABLE 4
Divisors of level-3 projective functions with $\mathbb{Z}(\zeta_3)$ coefficients for $c \leqslant 2$

c	$f=(p_0,\ p_1,\ p_{-1},\ p_{i\infty})$	$h \mod \frac{1}{3}$
$\frac{2}{3}$	$(-\frac{1}{12},\ -\frac{1}{12},\ -\frac{1}{12},\ \frac{1}{4})$	$0,\ \frac{1}{9}\cdot$
$\frac{4}{3}$	$(-\frac{1}{6},\ -\frac{1}{6},\ -\frac{1}{6},\ \frac{1}{2})$	$0,\ \frac{2}{9}\cdot$
$\frac{4}{3}$	$(-\frac{1}{6},\ -\frac{1}{6},\ \frac{1}{6},\ \frac{1}{6})$	$0,\ \frac{1}{9}\cdot$
2	$(-\frac{1}{4},\ -\frac{1}{4},\ -\frac{1}{4},\ \frac{3}{4})$	$0\cdot$
2	$(-\frac{1}{4},\ -\frac{1}{4},\ \frac{1}{12},\ \frac{5}{12})$	$0,\ \frac{1}{9},\ \frac{2}{9}\cdot$
2	$f_7=(-\frac{1}{4},\ \frac{1}{12},\ \frac{1}{12},\ \frac{1}{12})$	$0,\ \frac{1}{9}\cdot$

of level-3 cusp functions

$$f \propto \eta_0^{3p_0}\eta_1^{3p_1}\eta_{-1}^{3p_{-1}}\eta_{i\infty}^{3p_{i\infty}}. \tag{7.12}$$

That the differences of the exponents be integral implies that $p_\nu \in \frac{1}{12}\mathbb{Z}$ and $p_\nu - p_{\nu'} \in \frac{1}{3}\mathbb{Z}$, which constrains $c \in \frac{2}{3}\mathbb{Z}$ and $h \in \frac{1}{9}\mathbb{Z}$. Divisors satisfying the integrality condition for $c \leqslant 2$ are given in table 4.

Finally, the cusp representation of a level-5 projective modular function is

$$f = \prod_{\{c\}} \eta_c^{p_c + p_{c'} + \sum_{i=1}^{5} p_{c_i}}, \tag{7.13}$$

where c' is the cusp opposite c, and c_i are the five nearest cusps, where we are describing the cusps in terms of their images under h_5, i.e., as vertices of an icosahedron. For f to have coefficients in $\mathbb{Z}(\zeta_5)$, we must have $p_\nu \in \frac{1}{60}\mathbb{Z}$ and $p_\nu - p_{\nu'} \in \frac{1}{5}\mathbb{Z}$, whence $c \in \frac{2}{25}\mathbb{Z}$ and $h \in \frac{1}{25}\mathbb{Z}$. We refrain from giving a table of the allowed divisors because the list grows quickly for interesting values of c (i.e., $c > 1$).

Merely knowing the modular invariants out of which $\Gamma(N)$-factorizable partition functions must be composed is not as helpful for constructing candidate partition functions as might be wished, because we cannot yet determine, in general, which linear combinations of these invariants satisfy R_1 – R_5. In specific cases, certain results can be deduced. For example, one can show that partition functions at $c = 1$ which factorize on $\Gamma(4)$ and which satisfy R_2 and R_3 must be linear combinations of the form $Z = \frac{1}{4}\sum_{n=1}^{4} \mathscr{Y}_{i_n}$ where the $\mathscr{Y}_{i_n}$ belong to the set $\{\mathscr{Y}_2, \mathscr{Y}_3, \mathscr{Y}_5, \mathscr{Y}_6\}$. Indeed, all of the known $c = 1$ $\Gamma(4)$-factorizable theories are of this form, as we have seen above, but apart from trial and error, we do not know whether other combinations of this form also satisfy the physical requirements which we impose on acceptable partition functions, and may even correspond to new conformal field theories.

Clearly, more powerful methods are needed before we have an effective tool in exploring for conformal field theories.

8. Conclusions and prospect

We have considered rational conformal field theories whose one-loop partition function factorizes on one of the normal genus-zero subgroups Γ' of the modular group, viz., Γ, Γ^2, Γ^3, $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$ and $\Gamma(5)$. We have shown that the one-loop partition function is a finite sum of modular invariant expressions, each of which can be associated with a pair of projective modular functions of Γ'. By using the fact that the fundamental region of these subgroups is a genus-zero Riemann surface and therefore that modular functions of them belong to a genus-zero function field, and by imposing very general physical conditions on the resulting partition function, we have been able to narrow down these projective modular functions immensely.

For Γ (and, therefore, for Γ^2 and Γ^3), the only allowed partition functions for $c \leqslant 24$ must be composed of $j^{1/3}$, $j^{2/3}$ and $j + a$, $a \in \mathbb{Z}$, $-744 \leqslant a \leqslant 196139$. For the principal congruence subgroups of level $N \leqslant 5$, and for $c \geqslant 24/N$, there are *a priori* many modular invariants which could comprise the partition function. For c less than this critical value, however, there exists only a small finite set of projective modular functions from which we can form allowable modular invariants, and we have shown how to find and write these down explicitly in terms of a set of basis functions which we call cusp functions. Moreover, for conformal field theories which factorize on these subgroups, we have derived the following constraints on the conformal charges and weights:

$$\begin{aligned} &N = 2, \quad c \in \tfrac{1}{2}\mathbb{Z}, \quad h \in \tfrac{1}{16}\mathbb{Z}; \\ &N = 3, \quad c \in \tfrac{2}{3}\mathbb{Z}, \quad h \in \tfrac{1}{9}\mathbb{Z}; \\ &N = 4, \quad c \in \tfrac{1}{4}\mathbb{Z}, \quad h \in \tfrac{1}{32}\mathbb{Z}; \\ &N = 5, \quad c \in \tfrac{2}{25}\mathbb{Z}, \quad h \in \tfrac{1}{25}\mathbb{Z}. \end{aligned} \tag{8.1}$$

How might these results be extended to other subgroups? First, we might consider genus-zero subgroups that are not normal. For example, the subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c = 0 \bmod N \right\}, \tag{8.2}$$

though of smaller index than $\Gamma(N)$ for a given value of N, have fundamental

regions of genus zero for many values of $N \geqslant 6$ [11]. For these subgroups, hauptfunktions can be found, and the argument made above that projective modular functions are linear combinations of restricted projective functions probably goes through. Restricted projective modular functions of $\Gamma_0(N)$ can be written in terms of generalized eta functions of level N since $\Gamma_0(N) \supset \Gamma(N)$. On the other hand, unlike $\mathscr{H}/\Gamma(N)$, $\mathscr{H}/\Gamma_0(N)$ has orbifold points at which fractional zeros may be placed without violating holomorphicity. This will enlarge the space of allowed modular functions somewhat.

Principal congruence subgroups of level $N \geqslant 6$ do not have genus-zero fundamental regions, so we can no longer write an arbitrary modular function of level N as a rational function of a single hauptfunktion. An arbitrary modular function still has a cusp expansion in $q^{1/N}$, however, so the conformal charge of a modular invariant made from a modular function (without multipliers) must be a multiple of $24/N$. As before, we must also consider modular invariants formed out of projective modular functions (i.e., with multipliers). Now, any *restricted* projective modular function can be written as a product of powers of generalized eta functions of level N, or equivalently, as a product of powers of cusp functions of level N and of levels N' for all $N'|N$. For the function to have coefficients in $\mathbb{Z}(\zeta_N)$, the powers of the cusp functions must be integral, and by considering their leading powers of q, we may show that the resulting modular invariant has $c \in (2/N^2)\mathbb{Z}$, $h \in (1/N^2)\mathbb{Z}$ for odd N, $c \in (4/N^2)\mathbb{Z} \oplus (1/2)\mathbb{Z}$, $h \in (1/2N^2)\mathbb{Z} \oplus (1/16)\mathbb{Z}$ for even N, $c \in (6/N^2)\mathbb{Z}$, $h \in (1/N^2)\mathbb{Z}$ for $3|N$ and N odd, and $c \in (12/N^2)\mathbb{Z} \oplus (1/2)\mathbb{Z}$, $h \in (1/2N^2)\mathbb{Z} \oplus (1/16)\mathbb{Z}$ for $6|N$. This would be a very interesting result were it to hold for modular invariants formed from arbitrary projective modular functions, as it would bear out the conjecture (see, e.g., ref. [5]) that the denominators of c and h increase with the index of the subgroup, and demonstrate the exact rate of increase. Unfortunately, the arguments given above that (i) projective modular functions appearing in rational conformal field theories have rational phases, and that (ii) projective modular functions with rational phases and holomorphic in the upper half plane are linear combinations of restricted projective modular functions, both relied on the genus-zero nature of the function field and are not valid for general N.

Nonetheless, in a number of known rational conformal field theories, the partition functions are made from restricted projective modular functions. Itzykson and Zuber have written the partition functions of the minimal models with $c = 1 - 6/m(m+1)$ in terms of generalized eta functions of level $m(m+1)$ [18]. Also, the partition function for a single free scalar field on a circle with $R^2 = \frac{1}{2}(p/q)$, where $p, q \in \mathbb{Z}$ is expressible in terms of $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (2pq\tau)/\eta(\tau)$, $a \in (1/2pq)\mathbb{Z}$ [9], and using the Jacobi triple product identity, one can write this as a product of generalized eta functions of level $4pq$. Thus, there seems to be some evidence for the conjecture that the projective modular functions which constitute the partition functions of rational conformal field theories are linear combinations of restricted projective modular functions with rational phases.

Let us just mention what prevents argument (i) from going through in the case $N \geqslant 6$. The number of independent phases of the multipliers of a projective modular function is equal to the number of generators of the (free) group $\Gamma(N)$, viz., $\sigma_\infty - 1 + 2g$. The first $\sigma_\infty - 1$ generators identify the pairs of boundaries of the fundamental region which are attached to the same cusp, and the rationality of the phases associated with these generators follows from the rationality of c and h. After the identifications under these generators have been made, we are left with a polygon, and the remaining $2g$ generators identify pairs of sides of this polygon, resulting in a genus-g Riemann surface. It is the rationality of the phases associated with these $2g$ generators that is in question for $N \geqslant 6$.

In fact, the question of whether (ii) holds for $N \geqslant 6$ also strongly depends on the values of these additional $2g$ phases. As a concrete example, consider $\Gamma(6)$, whose fundamental region is a Riemann surface of genus 1 and modular parameter $\tilde{\tau} = e^{i\pi/3}$. (We mark the modular parameter with a tilde to distinguish it from τ, which is the coordinate on the Riemann surface itself.) The jacobian variety for this surface is the complex plane (with complex parameter ν) modulo the lattice generated by 1 and $\tilde{\tau}$. Since $g = 1$, the points of the Riemann surface, parametrized by τ with the identifications $\tau \sim A\tau$, $\forall A \in \Gamma(6)$, are in one-to-one correspondence with the points of the jacobian variety under the Abel map $\nu(\tau) = \int_{i\infty}^{\tau} \omega$, where ω is the abelian differential of the first kind on the surface. Furthermore, we know that $\omega \propto \eta^4(\tau)\, d\tau$ because $\eta^4(\tau)$ is the entire modular form of level 6 and weight 2 which vanishes at all the cusps. Using this fact, we learn that the images of the 12 cusps of $\mathcal{F}_6$ in the jacobian variety constitute the sublattice generated by $\frac{1}{6} + \frac{1}{6}\tilde{\tau}$ and $\frac{1}{3} - \frac{1}{6}\tilde{\tau}$. Now, a projective modular function $f(\tau)$ of level 6 has rational phases under the first $\sigma_\infty - 1 = 11$ generators and therefore some power of this function, $F(\tau)$, has no phases under these generators. It may still have phases under the last $2g = 2$ generators, which produce the identifications $\nu \sim \nu + 1$, $\nu \sim \nu + \tilde{\tau}$. Viewed as a function on the jacobian variety, $F(\nu, \tilde{\tau}) \equiv F(\tau(\nu))$ is therefore a projective elliptic function, doubly periodic up to phases: $F(\nu + 1, \tilde{\tau}) = e^{2\pi i\alpha} F(\nu, \tilde{\tau})$, $F(\nu + \tilde{\tau}, \tilde{\tau}) = e^{2\pi i\beta} F(\nu, \tilde{\tau})$. Abel's theorem informs us, however, that the phases α and β depend on the locations of the zeros and poles of the function. In particular, in order for F to be a restricted modular function, α and β must be sixth roots of unity, and therefore the phases of f under the last two generators must be related to its phases under the first eleven. In general, for $N \geqslant 6$, if we are to write a projective modular function in terms of restricted projective functions, i.e., cusp functions, we must show that the phases associated with the last $2g$ generators are rational and are related in a precise way to the other $\sigma_\infty - 1$ phases.

Requirement R_3 of the integrality of the coefficients of the partition function enabled us to argue for $N \leqslant 5$ that f must be the product of *integral* powers of cusp functions. We believe that R_3 is a strong enough restriction to require, in addition, that (i) and (ii) hold for $N \geqslant 6$. The proof of this, however, is likely to be rather involved.

Another pertinent and challenging task is to find general criteria for forming linear combinations of modular invariants into candidate partition functions which satisfy the physical requirements which we set out above. One approach would be to ask whether the projective modular functions which appear in the partition function of a rational conformal field theory are the characters of some chiral algebra. The theory would then only contain a finite number of representations of this algebra. Moreover, the transformation properties of these characters under modular transformations could give information about the operator product coefficients of the algebra [19]. If we can determine which linear combinations of modular invariants satisfy our requirements, we will be able to ask whether such partition functions correspond to actual conformal field theories. If so, we will have found a powerful method for finding new conformal field theories. More likely, we may find that additional restrictions need to be imposed, perhaps in the form of consistency at higher loops, which we have not considered in this paper.

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Appendix A

THETA FUNCTIONS AND ETA FUNCTIONS

In this appendix, we collect various definitions and formulae which are used in this paper. A theta function with characteristics a and b is defined as

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+a)^2 + 2\pi i n b}. \tag{A.1}$$

The Jacobi theta functions are

$$\begin{aligned}
\vartheta_2(\tau) &= \vartheta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2/2} = 2q^{1/12}\eta(\tau)\prod_{n=1}^{\infty}(1+q^n)^2, \\
\vartheta_3(\tau) &= \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2} = q^{-1/24}\eta(\tau)\prod_{n=0}^{\infty}(1+q^{n+1/2})^2, \\
\vartheta_4(\tau) &= \vartheta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(\tau) = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2/2} = q^{-1/24}\eta(\tau)\prod_{n=0}^{\infty}(1-q^{n+1/2})^2,
\end{aligned} \tag{A.2}$$

where $q = e^{2\pi i\tau}$, and

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{A.3}$$

These satisfy

$$\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3. \tag{A.4}$$

Under the generators of Γ, the Jacobi theta functions transform into one another as

$$\begin{aligned}
&\vartheta_2(\tau+1) = e^{\pi i/4}\vartheta_2(\tau), && \vartheta_2(-1/\tau) = (-i\tau)^{1/2}\vartheta_4(\tau),\\
&\vartheta_3(\tau+1) = \vartheta_4(\tau), && \vartheta_3(-1/\tau) = (-i\tau)^{1/2}\vartheta_3(\tau),\\
&\vartheta_4(\tau+1) = \vartheta_3(\tau), && \vartheta_4(-1/\tau) = (-i\tau)^{1/2}\vartheta_2(\tau).
\end{aligned} \tag{A.5}$$

The generalized Dedekind eta functions [11] have the product expansions

$$\eta^{(N)}_{(g,h)}(\tau) = \alpha_0(g,h)\,q^{P_2(g/N)/2}\prod_{\substack{n>0\\ n=g \bmod N}}\left(1-\zeta_N^h q^{n/N}\right)\prod_{\substack{n>0\\ n=-g \bmod N}}\left(1-\zeta_N^{-h}q^{n/N}\right),$$

$$\alpha_0(g,h) = \begin{cases}\left(1-\zeta_N^{-h}\right)e^{i\pi P_1(h/N)} = \left|2\sin\left(\dfrac{\pi h}{N}\right)\right|, & \text{if } g=0,\ h\neq 0 \bmod N,\\ 1, & \text{otherwise},\end{cases} \tag{A.6}$$

where $g, h \in \mathbb{Z}_N$, $\zeta_N = e^{2\pi i/N}$, and

$$\begin{aligned}
P_1(x) &= x - [x] - \tfrac{1}{2},\\
P_2(x) &= (x-[x])^2 - (x-[x]) + \tfrac{1}{6},
\end{aligned} \tag{A.7}$$

are the first and second Bernoulli polynomials. It is easy to verify that $\eta_{(g,h)} = \eta_{(-g,-h)}$. Note that $\eta^{(N)}_{(0,0)} = \eta^2$ is actually a projective modular form of unit weight. The rest of the $\eta^{(N)}$'s are projective modular functions and transform into each other up to a phase under modular transformations

$$\eta_{(g,h)}\left(\frac{a\tau+b}{c\tau+d}\right) \propto \eta_{(ag+ch,\,bg+dh)}(\tau), \qquad (g,h)\neq(0,0). \tag{A.8}$$

In sect. 7, we introduced the cusp functions $\eta_c^{(N)}(\tau)$. We define these in terms of the generalized eta functions as follows

$$\eta_0^{(2)} = \sqrt{\eta_{(1,0)}^{(2)}} = \sqrt{\vartheta_4/\eta}\,,$$

$$\eta_1^{(2)} = \sqrt{\eta_{(1,1)}^{(2)}} = \sqrt{\vartheta_3/\eta}\,,$$

$$\eta_{i\infty}^{(2)} = \sqrt{\eta_{(0,1)}^{(2)}} = \sqrt{\vartheta_2/\eta}\,. \tag{A.9}$$

$$\eta_0^{(3)} = \eta_{(1,0)}^{(3)}\,, \qquad \eta_1^{(3)} = \eta_{(1,2)}^{(3)}\,,$$

$$\eta_{-1}^{(3)} = \eta_{(1,1)}^{(3)}\,, \qquad \eta_{i\infty}^{(3)} = \eta_{(0,1)}^{(3)}\,. \tag{A.10}$$

$$\eta_0^{(4)} = \eta_{(1,0)}^{(4)}\,, \qquad \eta_2^{(4)} = \eta_{(1,2)}^{(4)}\,,$$

$$\eta_1^{(4)} = \eta_{(1,3)}^{(4)}\,, \qquad \eta_{-1}^{(4)} = {}^{(4)}_{(1,1)}\,,$$

$$\eta_{1/2}^{(4)} = \eta_{(2,1)}^{(4)}\,, \qquad \eta_{i\infty}^{(4)} = \eta_{(0,1)}^{(4)}\,. \tag{A.11}$$

$$\eta_0^{(5)} = \eta_{(1,0)}^{(5)}\,, \qquad \eta_{5/2}^{(5)} = \eta_{(2,0)}^{(5)}\,,$$

$$\eta_4^{(5)} = \eta_{(1,1)}^{(5)}\,, \qquad \eta_{9/2}^{(5)} = \eta_{(2,1)}^{(5)}\,,$$

$$\eta_3^{(5)} = \eta_{(1,2)}^{(5)}\,, \qquad \eta_{3/2}^{(5)} = \eta_{(2,2)}^{(5)}\,,$$

$$\eta_2^{(5)} = \eta_{(1,3)}^{(5)}\,, \qquad \eta_{7/2}^{(5)} = \eta_{(2,3)}^{(5)}\,,$$

$$\eta_1^{(5)} = \eta_{(1,4)}^{(5)}\,, \qquad \eta_{1/2}^{(5)} = \eta_{(2,4)}^{(5)}\,,$$

$$\eta_{i\infty}^{(5)} = \eta_{(0,1)}^{(5)}\,, \qquad \eta_{2/5}^{(5)} = \eta_{(0,2)}^{(5)}\,. \tag{A.12}$$

The product of all the cusp functions is always a constant

$$\eta_0^{(2)}\eta_1^{(2)}\eta_{i\infty}^{(2)} = \sqrt{2}\,,$$

$$\eta_0^{(3)}\eta_1^{(3)}\eta_{-1}^{(3)}\eta_{i\infty}^{(3)} = \sqrt{3}\,,$$

$$\eta_0^{(4)}\eta_2^{(4)}\eta_1^{(4)}\eta_{-1}^{(4)}\eta_{1/2}^{(4)}\eta_{i\infty}^{(4)} = \sqrt{2}\,,$$

$$\eta_0^{(5)}\eta_1^{(5)}\eta_2^{(5)}\eta_3^{(5)}\eta_4^{(5)}\eta_{i\infty}^{(5)}\eta_{5/2}^{(5)}\eta_{9/2}^{(5)}\eta_{3/2}^{(5)}\eta_{7/2}^{(5)}\eta_{1/2}^{(5)}\eta_{2/5}^{(5)} = \sqrt{5}\,. \tag{A.13}$$

In terms of their product representations, these equations are, in a sense, generalizations of the Euler identity $\prod_{n=1}^{\infty}(1+q^n)(1-q^{2n-1})=1$.

Appendix B

ORBIFOLD PARTITION FUNCTIONS

The reader may have noticed that the generalized eta functions bear a strong resemblance to the partition functions associated with orbifold compactifications [20]. Here we will make this connection precise.

Consider a set of complex bosons ϕ_r, $r=1,\ldots,p$, whose target space is the torus $\mathbb{C}^p/\Lambda$ is a rank $2p$ lattice. If this lattice admits the N-fold symmetry generated by θ: $\phi_r \to \theta_{rr'}\phi_{r'}$, then we can twist the theory by identifying points equivalent under this $\mathbb{Z}_N$ symmetry. We can diagonalize the twist so that $\theta_{rr'}=\mathrm{diag}(\zeta_N^{i_1},\ldots,\zeta_N^{i_p})$; thus, it suffices to consider a single complex boson twisted by θ: $\phi \to \zeta_N\phi$.

For a single complex boson, the partition function is

$$Z=\sum_{n=0}^{N-1}\mathrm{Tr}_{H_n}\left[\mathscr{P}q^{L_0}\bar{q}^{\bar{L}_0}\right], \tag{B.1}$$

where H_n denotes the Hilbert space of string states obeying the boundary condition $\phi(\sigma+\pi)=\zeta_N^n\phi(\sigma)$, and $\mathscr{P}=(1/N)\sum_{m=0}^{N-1}\theta^m$ is the projection operator onto $\mathbb{Z}_N$-invariant states. Writing

$$Z=\frac{1}{N}\sum_{n,m=0}^{N-1}Z_{n,m}, \qquad Z_{n,m}=\mathrm{Tr}_{H_n}\left[\theta^m q^{L_0}\bar{q}^{\bar{L}_0}\right], \tag{B.2}$$

we see that $Z_{0,0}$ is the original partition function, and the remaining piece, $\tilde{Z}=\sum_{(n,m)\neq(0,0)}Z_{n,m}$, arises from the orbifold identification. It is the latter which we wish to calculate.

First consider the contribution coming from the untwisted sector, $Z_{0,m}$. θ^m acts on creation operators of positive moding by $\alpha_l^\dagger \to \zeta_N^{-m}\alpha_l^\dagger$, $\tilde{\alpha}_l^\dagger \to \zeta_N^{-m}\tilde{\alpha}_l^\dagger$, $l\in\mathbb{Z}^+$, whereas for negatively moded creation operators, $\alpha_l \to \zeta_N^m\alpha_l$, $\tilde{\alpha}_l \to \zeta_N^m\tilde{\alpha}_l$, $l\in\mathbb{Z}^-$. Thus, its action on states in H_0 is given by

$$\begin{aligned}\theta^m\colon\ &\mathscr{A}\left(N_+,N_-,\tilde{N}_+,\tilde{N}_-\right)|p_{\mathrm{L}},p_{\mathrm{R}}\rangle \\ &\to \zeta_N^{m(-N_++N_--\tilde{N}_++\tilde{N}_-)}\mathscr{A}\left(N_+,N_-,\tilde{N}_+,\tilde{N}_-\right)|\zeta_N^m p_{\mathrm{L}},\zeta_N^m p_{\mathrm{R}}\rangle,\end{aligned} \tag{B.3}$$

where $\mathscr{A}(N_+,N_-,\tilde{N}_+,\tilde{N}_-)$ denotes a product of N_+ and $\tilde{N}_+$ positively moded and N_- and $\tilde{N}_-$ negatively moded creation operators. Since $\langle p_{\mathrm{L}},p_{\mathrm{R}}|\zeta_N^m p_{\mathrm{L}},\zeta_N^m p_{\mathrm{R}}\rangle$ vanishes unless $p_{\mathrm{L}}=p_{\mathrm{R}}=0$, the trace is only over states consisting of creation

operators acting on $|0,0\rangle$. After including the correct normal ordering constant in L_0, $\bar{L}_0$, viz., $-\frac{1}{24}c = -\frac{1}{12}$, we obtain (for $m \neq 0$)

$$Z_{0,m} = q^{-1/12}\bar{q}^{-1/12}\Bigg[\prod_{l_+=1}^{\infty}\left(1-\zeta_N^{-m}q^{l_+}\right)\prod_{l_-=1}^{\infty}\left(1-\zeta_N^{m}q^{l_-}\right) \times \prod_{\tilde{l}_+=1}^{\infty}\left(1-\zeta_N^{-m}\bar{q}^{\tilde{l}_+}\right)\prod_{\tilde{l}_-=1}^{\infty}\left(1-\zeta_N^{m}\bar{q}^{\tilde{l}_-}\right)\Bigg]^{-1}, \tag{B.4}$$

which is equal to $|\alpha_0(0,m)/\eta^{(N)}_{(0,m)}|^2$.

In the twisted sector H_n, the oscillators have non-integral moding, $\alpha_{l+n/N}$, $\tilde{\alpha}_{l-n/N}$, $l \in \mathbb{Z}$. This changes the normal-ordering constant to $\zeta(-1, n/N)$ where the Hurwitz zeta function $\zeta(s,a)$ is the analytic continuation of $\Sigma_{l=0}^{\infty}[1/(l+a)^s]$. This can be evaluated [21] to give $-\frac{1}{2}(n/N)^2 + \frac{1}{2}(n/N) - \frac{1}{12} = -\frac{1}{2}P_2(n/N)$, where $P_2(x)$ is the second Bernoulli polynomial (A.7). By reasoning similar to that above, we now obtain ($n \neq 0$)

$$\begin{aligned} Z_{n,m} &= \chi_{n,m}(q\bar{q})^{-P_2(n/N)/2}\Bigg[\prod_{l_+=0}^{\infty}\left(1-\zeta_N^{-m}q^{l_++n/N}\right)\prod_{l_-=1}^{\infty}\left(1-\zeta_N^{m}q^{l_--n/N}\right) \\ &\quad\times\prod_{\tilde{l}_+=1}^{\infty}\left(1-\zeta_N^{-m}\bar{q}^{\tilde{l}_+-n/N}\right)\prod_{\tilde{l}_-=0}^{\infty}\left(1-\zeta_N^{m}\bar{q}^{\tilde{l}_-+n/N}\right)\Bigg]^{-1} \\ &= \chi_{n,m}|\eta^{(N)}_{(n,-m)}|^{-2}, \end{aligned} \tag{B.5}$$

where $\chi_{n,m}$ is the multiplicity which depends on the number of fixed points. The number of fixed points of a rotation $\Theta \in \mathrm{SO}(2)$ of order k, $\Theta^k = 1$, is given by the Lefschetz fixed-point theorem as $\det(1-\Theta) = 2(1-\cos(2\pi/k)) = 4\sin^2(\pi/k) = \alpha_0^2(0, N/k)$, where $\alpha_0(g,h)$ is defined in eq. (A.6). States in H_n are closed up to identification by a group element of order $k = N/\langle n, N\rangle$ (where $\langle n, N\rangle$ denotes the greatest common divisor of n and N), and so must be centered about one of the $\alpha_0^2(0,\langle n,N\rangle)$ fixed points of order k. Under θ^m, however, the center-of-mass coordinate $x \to \zeta_N^m x$, so unless x is a fixed point of order $N/\langle m, N\rangle$ as well, the contribution to the trace (B.2) of the states in H_n associated with that fixed point will vanish because $\langle x|\zeta_N^m x\rangle = 0$ for $x \neq \zeta_N^m x \bmod \Lambda$. Thus, x must be a fixed point of order $N/\langle n, m, N\rangle$, whence the number of fixed points contributing to $Z_{n,m}$ is $\chi_{n,m} = \alpha_0^2(0,\langle n,m,N\rangle)$, and $Z_{n,m} = |\alpha_0(0,\langle n,m,N\rangle)/\eta^{(N)}_{(n,-m)}|^2$.

In the case where there is only one complex boson from the start, N must be 2, 3, 4 or 6, so that $\alpha_0(0,m) = \alpha_0(0,\langle 0,m,N\rangle)$, and the total contribution to the

partition function due to the orbifold identification is

$$\tilde{Z} = \sum_{(n,m)\neq(0,0)} \left| \frac{\alpha_0(0, \langle n, m, N \rangle)}{\eta^{(N)}_{(n,-m)}} \right|^2 . \tag{B.6}$$

Note that the factor in the numerator is the same for all (n, m) belonging to a common modular orbit, insuring that $\tilde{Z}$ is modular invariant, since the $\eta^{(N)}$'s simply transform into one another up to a phase under modular transformations. For N prime, there is only one modular orbit, so for $N = 3$, for example, $\tilde{Z} = 3\Sigma_{(n,m)\neq(0,0)}|1/\eta^{(3)}_{(n,m)}|^2 \propto Z(f_7, f_7)$ (see table 4).

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