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MASS RELATIONS AND NEUTRINO OSCILLATIONS IN AN SO(10) MODEL

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We present in detail a grand unified model based on SO(10) which naturally reproduces known fermion mass relations and mixing angles. It predicts $m_{\nu_e} \ll m_{\nu_\mu}, m_{\nu_\tau}$, and predominant ν_μ-ν_τ oscillations. Several other features of the model are analyzed in detail: construction of the Higgs potential, predictions for τ_p and $\sin^2\theta_w$ in a temporarily free model and a relation between the mass of the t-quark and the lifetime of the B meson.

1. Introduction

In a previous paper [1] we presented a grand unified model based on the gauge group SO(10) which reproduced the observed fermion masses and mixing angles in a technically natural way at the cost of introducing a large number of fundamental Higgs particles. In this paper we will study the implications of this model in further detail.

Perhaps the strongest signal of unification beyond SU(5) would be the observation of neutrino oscillations. These are forbidden in standard SU(5) as a result of the global $B-L$ symmetry, but are allowed in SO(10) where $B-L$ is violated. In SO(10) models the masses and mixings in the neutrino sector are related to those in the charge $\frac{2}{3}$ quark sector. Neutrino oscillations thus provide an important test both of the idea of SO(10) grand unification and of relations between masses and mixing

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angles. In our models the forms of the mass matrices with $\Delta I_w = \frac{1}{2}$ and $\Delta I_w = 0$ allow us to make a number of predictions for neutrino masses and oscillations which depend only on the top quark mass and on a single mass ratio appearing in the $\Delta I_w = 0$ sector. In all cases we find $m_{\nu_e} \ll m_{\nu_\mu}, m_{\nu_\tau}$ and appreciable mixing only between ν_μ and ν_τ.

Realistic fits to the observed fermion mass spectrum in grand unified models seem to require quite complicated Higgs sectors [1-3]. We consider two problems associated with models with a large number of Higgs bosons. The first is the effect of Higgs bosons on the evolution of the gauge coupling constants and hence on the predictions for the weak mixing angle, θ_w, and the proton lifetime. In models such as the one presented here, asymptotic freedom is lost at some scale due to the large number of scalar bosons. We find that reasonable assumptions for the distribution of scalar masses allow us to find acceptable fits to both $\sin^2\theta_w$ and the proton lifetime while keeping all couplings perturbatively small below the Planck mass. A large number of Higgs fields also makes direct minimization of the Higgs potential difficult. If the predictions of the model depend on the assumed form of the vacuum expectation values, then one must show that this set of expectation values is realized for a finite range of parameters in the Higgs potential. We show how this may be done and verify the naturalness of our choice of vacuum expectation values to lowest order in the gauge hierarchy.

The outline of this paper is the following. In sect. 2 we review some salient features of SO(10) and present our model. In sect. 3 we analyze the charged fermion masses and mixing angles and then study the consequences of these masses and mixings for the neutral fermion sector. We find stringent bounds on the allowed neutrino masses if our model is to be consistent with recent experimental bounds on the lack of ν_μ-ν_τ oscillations. In sect. 4 we construct the Higgs potential for our model and study the naturalness of the Higgs vacuum expectation values. In sect. 5 we consider the evolution of the coupling constants for a particular breaking scheme and find that $\sin^2\theta_w$ and the proton lifetime may be consistent with present observations even if asymptotic freedom is eventually lost. Details of the spinor representations and couplings in SO(N) groups are given in the appendix.

2. Description of the model

The group SO(10) appears [4] as a natural generalization of the unifying gauge group SU(5) and of the effective low-energy gauge group $SU(2)_L \times U(1) \times SU(3)_c$, where $SU(3)_c$ is the color group and $SU(2)_L \times U(1)$ is the electroweak group of the Weinberg-Salam-Ward-Glashow model. To elicit the structure of SO(10) it will often be useful to decompose it with respect to SU(5):

$$SO(10) \supset SU(5) \times U(1), \tag{2.1}$$

while the weak isospin group $SU(2)_L$ appears in the chiral decomposition

$$SO(10) \supset SU(2)_L \times SU(2)_R \times SU(4)\,; \tag{2.2}$$

here SU(4) is the Pati-Salam generalized color

$$SU(4) \supset SU(3)_c \times U(1)\,. \tag{2.3}$$

The SO(10) group is automatically anomaly free since it has no d-coefficient which is another way of saying that the symmetric product of two adjoint representations does not contain the adjoint representation.

The left-handed spin $\frac{1}{2}$ fermions are assigned to the fundamental 16-dimensional complex spinor representation of SO(10). Each such representation describes one family of fermions. Under the chiral decomposition (2.2) we have

$$\underset{\sim}{16} = (\underset{\sim}{2}, \underset{\sim}{1}, \underset{\sim}{4}) + (\underset{\sim}{1}, \underset{\sim}{2}, \underset{\sim}{\bar{4}})\,, \tag{2.4}$$

where the first $SU(2)_L$ entry is the electroweak one. We obtain the color content from (2.3)

$$\underset{\sim}{16} = (\underset{\sim}{2}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{2}, \underset{\sim}{1}, \underset{\sim}{3}^c) + (\underset{\sim}{1}, \underset{\sim}{2}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{2}, \underset{\sim}{\bar{3}}^c)\,. \tag{2.5}$$

We identify the charge operator with

$$Q = I_{3L} + I_{3R} + \frac{2}{\sqrt{6}} I_{15}\,; \tag{2.6}$$

it lies entirely in SU(5), and

$$\mathrm{Tr}\, I_{3L}^2 = \mathrm{Tr}\, I_{3R}^2 = \mathrm{Tr}\, I_{15}^2 = \tfrac{1}{2}\,, \tag{2.7}$$

where I_{15} denotes the 15th generator of the SU(4) with value $1/2\sqrt{6}$ for leptons. It is proportional to $B - L$ which is seen to be gauged in SO(10). With this assignment it is easy to see that each family of fermions contains one charge $\frac{2}{3}$ quark, one charge $-\frac{1}{3}$ quark, one charge -1 lepton, each appearing in their left- and right-handed modes; it also contains two left-handed neutral leptons. The presence of this extra neutral lepton degree of freedom can be seen in the SU(5) decomposition of a family

$$\underset{\sim}{16} = \underset{\sim}{\bar{5}} + \underset{\sim}{10} + \underset{\sim}{1}\,. \tag{2.8}$$

This extra neutral degree of freedom per family, not contained in the SU(5) model, is a characteristic signature of SO(10); it will lead to lepton number violation and hence to neutrino oscillations.

In this model we will have three fermion families. The first one contains (neglecting for the moment mixings) the u, d quarks, the electron e^-, its neutrino ν_e, and a new neutral degree of freedom N_e which will have to be sufficiently massive to have escaped detection. The second family is comprised of the c, s quarks, the muon μ, its neutrino ν_μ and N_μ. The third family consists of the t, b quarks, the tau τ^- lepton, its neutrino ν_τ, and N_τ. Of all these particles, the t-quark, and N_e, N_μ, N_τ leptons have not yet been observed. The mass of the t-quark will be related to the width of the b-quark, yielding only an upper bound, given the present experimental situation. This particular choice of fermion representations offers no explanation for the triplication of families – it is just taken as a fact (assuming the existence of the t-quark).

The vector particles of the theory fill the 45-dimensional representation of SO(10); under the chiral decomposition (2.1) they appear as

$$\underset{\sim}{45} = (\underset{\sim}{3},\underset{\sim}{1},\underset{\sim}{1}) + (\underset{\sim}{1},\underset{\sim}{3},\underset{\sim}{1}) + (\underset{\sim}{1},\underset{\sim}{1},\underset{\sim}{15}) + (\underset{\sim}{2},\underset{\sim}{2},\underset{\sim}{6})\,; \tag{2.9}$$

the further decompositions

$$\underset{\sim}{15} = \underset{\sim}{8}^c + \underset{\sim}{3}^c + \bar{\underset{\sim}{3}}^c + \underset{\sim}{1}^c\,, \tag{2.10}$$

$$\underset{\sim}{6} = \underset{\sim}{3}^c + \bar{\underset{\sim}{3}}^c \tag{2.11}$$

yield the physical content: we have the usual $SU(2)_L$ triplet of charge ± 1, 0 of the electroweak model, plus its right-handed counterpart, and a color singlet neutral vector; the hypercharge-carrying vector of the electroweak model is identified with a linear combination of this neutral vector and the third component of the right-handed triplet, as shown by the choice of the charge operator (2.6). So, in the color neutral sector the model contains, beyond the electroweak vectors, a new charged vector and a new neutral one. Their masses will have to be sufficiently large for them to have escaped detection. In the color sector we have the usual octet of spin 1 gluons, a flavor singlet but color triplet (antitriplet) vector of charge $\frac{2}{3}$ $(-\frac{2}{3})$ and two weak doublets color triplets vectors of charge $(\frac{4}{3},\frac{1}{3})$ and $(\frac{1}{3},-\frac{2}{3})$, together with their antitriplets. Alternatively one can use the SU(5) decomposition.

$$\underset{\sim}{45} = \underset{\sim}{24} + \underset{\sim}{1} + \underset{\sim}{10} + \overline{\underset{\sim}{10}}\,, \tag{2.12}$$

and knowledge of the SU(5) model to find that the weak doublet color triplet vectors with charges $(\frac{4}{3},\frac{1}{3})$ reside in the $\underset{\sim}{24}$ and are therefore the leptoquark-antidiquarks of the Georgi-Glashow model, while the doublet with charge $(\frac{1}{3},-\frac{2}{3})$ belong to the $\underset{\sim}{10}$ of SU(5); these cause transitions between the new fermion degree of freedom and ordinary quarks as well as additional leptoquark, antidiquark transitions.

The Higgs content of the model is dictated by two requirements – breaking SO(10) down to $SU(2)_L \times U(1) \times SU(3)_c$ and by the desire to reproduce the phenomenologically observed mass and mixing angle parameters of the fermions. The fermion mass-giving candidate Higgs appear in the product

$$\underset{\sim}{16} \times \underset{\sim}{16} = (\underset{\sim}{10} + \underset{\sim}{126})_S + (\underset{\sim}{120})_A, \tag{2.13}$$

where S (A) means the symmetric (antisymmetric) part. Since these are left-handed fermions, the spin 0 sector appears in the product that is symmetric in the Lorentz indices; in particular the 120 coupling exists only by antisymmetrizing in family space. The 10-dimensional representation of SO(10) is the real vector representation which decomposes as

$$\underset{\sim}{10} = (\underset{\sim}{2}, \underset{\sim}{2}, \underset{\sim}{1}) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{6}) \tag{2.14}$$

under the chiral breakup (2.2), and as

$$\underset{\sim}{10} = \underset{\sim}{5} + \bar{\underset{\sim}{5}} \tag{2.15}$$

under the SU(5) one. The 126-dimensional representation is a complex representation corresponding to a fifth rank totally antisymmetric tensor. Such a representation would ordinarily be real but for the 10-dimensional Levi-Civitá symbol which provides the conjugation operation. Indeed, the fifth rank totally antisymmetric tensor has 252 components, but we can project out of it the $\underset{\sim}{126}$ and $\overline{\underset{\sim}{126}}$ by means of the operations

$$\underset{\sim}{126} \sim (\delta_{\alpha\beta} + \varepsilon_{\alpha\beta}) T_\beta, \tag{2.16}$$

$$\overline{\underset{\sim}{126}} \sim (\delta_{\alpha\beta} - \varepsilon_{\alpha\beta}) T_\beta, \tag{2.17}$$

where α, β stands for a group of five totally antisymmetrized SO(10) indices. Its chiral content is given by

$$\underset{\sim}{126} = (\underset{\sim}{2}, \underset{\sim}{2}, \underset{\sim}{15}) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{6}) + (\underset{\sim}{3}, \underset{\sim}{1}, \underset{\sim}{10}) + (\underset{\sim}{1}, \underset{\sim}{3}, \overline{\underset{\sim}{10}}), \tag{2.18}$$

where the $\underset{\sim}{15}$ and $\underset{\sim}{6}$ of SU(4) decompose as in (2.10) and (2.11) and

$$\underset{\sim}{10} = \underset{\sim}{1}^c + \underset{\sim}{3}^c + \underset{\sim}{6}^c. \tag{2.19}$$

We can see from (2.18) that the $\underset{\sim}{126}$ is not real since $\underset{\sim}{10} \neq \overline{\underset{\sim}{10}}$. Alternatively its SU(5) decomposition is

$$\underset{\sim}{126} = \underset{\sim}{1} + \bar{\underset{\sim}{5}} + \underset{\sim}{45} + \overline{\underset{\sim}{15}} + \underset{\sim}{50} + \underset{\sim}{10}. \tag{2.20}$$

From (2.18) and (2.20) we can read off the possible vacuum values the $\underset{\sim}{126}$ can take without violating color and charge. It is convenient to break these up with respect to their ΔI_w properties, where I_w denotes the weak isospin:

$$\Delta I_w = 0: \qquad \langle 126 \rangle \sim \begin{cases} (\underset{\sim}{1}, \underset{\sim}{3}, \underset{\sim}{1}^c) \text{ for chiral}, \\ \underset{\sim}{1} \text{ for SU(5)}, \end{cases} \tag{2.21}$$

$$\Delta I_w = \tfrac{1}{2}: \qquad \langle 126 \rangle \sim \begin{cases} (\underset{\sim}{2}, \underset{\sim}{2}, \underset{\sim}{1}^c) \text{ for chiral}, \\ \bar{\underset{\sim}{5}} \text{ and } \underset{\sim}{45} \text{ for SU(5)}, \end{cases} \tag{2.22}$$

$$\Delta I_w = 1: \qquad \langle 126 \rangle \sim \begin{cases} (\underset{\sim}{3}, \underset{\sim}{1}, \underset{\sim}{1}^c) \text{ for chiral}, \\ \overline{\underset{\sim}{15}} \text{ for SU(5)}. \end{cases} \tag{2.23}$$

Similarly the $\underset{\sim}{10}$ of SO(10) can only have $\Delta I_w = \frac{1}{2}$ vacuum values

$$\Delta I_w = \tfrac{1}{2}: \quad \langle \underset{\sim}{10} \rangle \sim \begin{cases} (\underset{\sim}{2}, \underset{\sim}{2}, \underset{\sim}{1}) \text{ for chiral}, \\ \underset{\sim}{5} \text{ and } \bar{\underset{\sim}{5}} \text{ for SU(5)}. \end{cases} \tag{2.24}$$

From the above we note that the only $\Delta I_w = 0$ value available to us is the SU(5) singlet in the $\underset{\sim}{126}$. This vev will clearly break SO(10) down to SU(5) but there is nothing left to break SU(5) down to $SU(2)_L \times U(1) \times SU(3)_c$. Hence we must add in the model some other scalar representation which does not couple to the fermions and yet does the required breaking. We have two likely candidates, namely the adjoint representation of SO(10) with the decompositions (2.9) and (2.12), and the "quadrupole" representation of SO(10) which is represented by a symmetric second-rank traceless tensor; it has 54 real components and breaks up chirally as

$$\underset{\sim}{54} = (\underset{\sim}{3}, \underset{\sim}{3}, \underset{\sim}{1}) + (\underset{\sim}{2}, \underset{\sim}{2}, \underset{\sim}{6}) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{20}) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{1}), \tag{2.25}$$

where

$$\underset{\sim}{20} = \underset{\sim}{8}^c + \underset{\sim}{6}^c + \bar{\underset{\sim}{6}}^c, \tag{2.26}$$

and

$$\underset{\sim}{54} = \underset{\sim}{24} + \underset{\sim}{10} + \overline{\underset{\sim}{10}} \tag{2.27}$$

under SU(5). Clearly the $\underset{\sim}{45}$ and $\underset{\sim}{54}$ both have the required $\Delta I_{\mathrm{w}}=0$ vacuum value

$$\Delta I_{\mathrm{w}}=0: \qquad \langle\underset{\sim}{54}\rangle \sim \begin{cases} (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{1}) \text{ for chiral}, \\ \underset{\sim}{24} \text{ for SU}(5), \end{cases} \tag{2.28}$$

$$\langle\underset{\sim}{45}\rangle \sim \begin{cases} (\underset{\sim}{1}, \underset{\sim}{3}, \underset{\sim}{1}^{\mathrm{c}}) \text{ and } (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{1}^{\mathrm{c}}) \text{ for chiral}, \\ \underset{\sim}{1} \text{ and } \underset{\sim}{24} \text{ for SU}(5). \end{cases} \tag{2.29}$$

The requirement of naturalness in our pattern of vacuum expectation values requires the use of the $\underset{\sim}{54}$ rather than the $\underset{\sim}{45}$, as pointed out to us by H. Georgi. Hence in the following we use the $\underset{\sim}{54}$ and discard the $\underset{\sim}{45}$. To summarize the situation: in order to effect the $\Delta I_{\mathrm{w}}=0$ breaking

$$\mathrm{SO}(10) \to \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1) \times \mathrm{SU}(3)_{\mathrm{c}}, \tag{2.30}$$

which entails a loss of two units of rank, we need two Higgs representations, the $\underset{\sim}{126}$ and the $\underset{\sim}{54}$ with

$$\langle\underset{\sim}{126}\rangle \sim \underset{\sim}{1} \text{ of SU}(5) \text{ or } (\underset{\sim}{1}, \underset{\sim}{3}, \underset{\sim}{1}^{\mathrm{c}}) \text{ chiral}, \tag{2.31}$$

$$\langle\underset{\sim}{54}\rangle \sim \underset{\sim}{24} \text{ of SU}(5) \text{ or } (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{1}^{\mathrm{c}}) \text{ chiral}. \tag{2.32}$$

The $\Delta I_{\mathrm{w}}=0$ breaking of the model is chosen out of convenience – just to get the job done. The $\Delta I_{\mathrm{w}}=\frac{1}{2}$ breaking is much more delicate since masses of quarks and leptons have that same quantum number. This mechanism must be chosen in a way that reproduces what is experimentally known about these parameters. In particular we wished to reproduce the Oakes relation

$$\tan^2\theta_{\mathrm{c}} = m_{\mathrm{d}}/m_{\mathrm{s}}, \tag{2.33}$$

the SU(5) relation [5]

$$m_{\mathrm{b}}/m_{\tau} = 1, \tag{2.34}$$

and the Georgi-Jarlskog relation [2]

$$m_{\mathrm{e}}/m_{\mu} \sim \tfrac{1}{9} m_{\mathrm{d}}/m_{\mathrm{s}}. \tag{2.35}$$

We find it possible to enforce these three relations in a way that is (technically) natural. The cost in terms of number of Higgs is high but we feel no great concern, especially since the nature of the Higgs particle is not settled; in particular, one can regard our model as a sophisticated phenomenological one. In order to achieve this $\Delta I_w = \frac{1}{2}$ pattern we need one complex Higgs transforming as a $\underset{\sim}{10}$ of SO(10) and two extra Higgs transforming as the $\underset{\sim}{126}$ of SO(10). Thus the particle content of our model is

spin 1: adjoint of vector bosons $\sim \underset{\sim}{45}$ of SO(10),

spin $\frac{1}{2}$: three families of $\underset{\sim}{16}$: $16_1, 16_2, 16_3$,

spin 0: real $\underset{\sim}{54}$, complex (doubled) $\underset{\sim}{10}$, three $\underset{\sim}{126}$'s: $126_1, 126_2, 126_3$.

Besides the Yang-Mills SO(10) couplings, we have the Yukawa couplings given by

$$\mathcal{L}_Y = (A16_1 \cdot 16_2 + B16_3 \cdot 16_3) \cdot \overline{126_1} + (a16_1 \cdot 16_2 + b16_3 \cdot 16_3) \cdot (10_1 + i10_2)$$
$$+ (c16_2 \cdot 16_2) \cdot \overline{126_2} + d16_2 \cdot 16_3 \cdot \overline{126_3} + \text{c.c.}, \tag{2.36}$$

where A, B, a, b, c, and d are undetermined Yukawa coupling constants. They are taken to be real in order to avoid hard (explicit) CP violation. In addition to SO(10) invariance, $\mathcal{L}_Y$ shows two continuous global phase symmetries, call these X and Y; they are summarized in the following table:

	16_1	16_2	16_3	$10_1 + i10_2$	126_1	126_2	126_3
X	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1	1	0
Y	1	-1	0	0	0	-2	-1.

(2.37)

When the Higgs acquire vacuum values, these phase symmetries will be spontaneously broken. Since they are not gauged, massless Goldstone-Nambu bosons will result. To avoid their presence we have to break X and Y by explicit terms in the Higgs potential in such a way as to preserve the naturalness of the Yukawa couplings. Remarkably enough this can be done because of a marvelous property of the complex $\underset{\sim}{126}$: the fully symmetrized fourfold product $(126)^4_S$ contains one SO(10) singlet,

$$(\underset{\sim}{126})^4_S = \underset{\sim}{1} + \cdots . \tag{2.38}$$

Thus we require that the Higgs potential contain terms like

$$V = \lambda_1 (126_1)^4 + \lambda_2 (126_2)^4 + \cdots + \text{c.c.} \tag{2.39}$$

The first term breaks X to a mod 4 discrete symmetry; the second one breaks Y to another mod 8 discrete symmetry. These two discrete symmetries suffice to keep the naturalness of the Yukawa couplings while avoiding the risk of massless Goldstone bosons. The rest of the Higgs self-couplings are selected so as to honor the remaining discrete symmetries. The detailed analysis of the Higgs self-couplings and their naturalness are treated separately in sect. 4. The Higgs are taken to develop the following vev's:

$$\Delta I_{\mathrm{w}}=0: \qquad \langle 54\rangle \sim \underset{\sim}{24} \text{ of SU(5) or } (\underset{\sim}{1},\underset{\sim}{1},\underset{\sim}{1}^{\mathrm{c}}) \text{ chiral},$$

$$\langle 126_1\rangle \sim \underset{\sim}{1} \text{ of SU(5) or } (\underset{\sim}{1},\underset{\sim}{2},\underset{\sim}{1}^{\mathrm{c}}) \text{ chiral}; \tag{2.40}$$

$$\Delta I_{\mathrm{w}}=\tfrac{1}{2}: \qquad \langle 126_2\rangle \sim \underset{\sim}{45} \text{ of SU(5) or } (\underset{\sim}{2},\underset{\sim}{2},\underset{\sim}{1}^{\mathrm{c}}) \text{ chiral},$$

$$\langle 126_3\rangle \sim \underset{\sim}{\bar{5}} \text{ of SU(5) or } (\underset{\sim}{2},\underset{\sim}{2},\underset{\sim}{1}^{\mathrm{c}}) \text{ chiral},$$

$$\langle 10_1+i10_2\rangle \sim \underset{\sim}{5}+\underset{\sim}{\bar{5}} \text{ of SU(5) or } (\underset{\sim}{2},\underset{\sim}{2},\underset{\sim}{1}^{\mathrm{c}}) \text{ chiral},$$

$$\langle 126_1\rangle \sim \underset{\sim}{\bar{5}} \text{ of SU(5) or } (\underset{\sim}{2},\underset{\sim}{2},\underset{\sim}{1}^{\mathrm{c}}) \text{ chiral}, \tag{2.41}$$

where for the complex $\underset{\sim}{10}$ the vev along the $\underset{\sim}{5}$ and the $\underset{\sim}{\bar{5}}$ need not be equal since the $\underset{\sim}{10}$ is doubled. That such a pattern of vev's can be naturally maintained will be demonstrated in sect. 4 to lowest order in the gauge hierarchy.

The detailed consequences of this model will be expounded in the following sections. For the moment let us enumerate without proof some of its salient features:

(a) In the fermion mass matrices, it reproduces the three relations (2.33)–(2.35); the mass of the t-quark is a free parameter but it turns out to be related to the lifetime of the B-meson by

$$\tau_{\mathrm{B}}=\left(\frac{m_{\mathrm{t}}}{m_{\mathrm{c}}}\right)4\times 10^{-15}\,\mathrm{sec}. \tag{2.42}$$

The b-quark decays via mixing in the KM matrix, predominantly to the c-quark. The six neutral leptons all acquire masses. Three of them are very massive, three are very light as in the Gell-Mann, Ramond, Slansky (G-MRS) mechanism [6]; appreciable neutrino oscillations are found to occur only in the ν_μ-ν_τ sector, with the consequence that our model predicts no presently detectable effect in reactor-induced neutrino oscillation experiments, as shown in sect. 3.

(b) The model has no globally conserved $B-L$ symmetry since Majorana masses are present. Thus it allows for some SU(5) forbidden decay modes of the proton.

(c) Due to the enormous number of Higgs particles the model is not asymptotically free but, rather, temporarily free. We do not consider this a handicap in view of our pragmatic attitude towards the Higgs, and the fact that the Landau "singularity" occurs at a distance smaller than the Planck length. We present in sect. 5 a detailed analysis of the coupling constant evolution equations; we find that the Weinberg angle does come out to be slightly larger than in the SU(5) model, in accordance with experiment.

3. Fermion masses and mixing angles

Since electric charge is unbroken it is convenient to separate the discussion of the fermion mass matrices according to the electric charge of the particles. Further, electromagnetic interactions conserve charge conjugation, C, with the result that the mass matrices of charged particles will necessarily be of the Dirac (lepton-number conserving) type, while the mass matrices of neutral particles can be of both of the Dirac and of the lepton-number violating Majorana types. So we start with the simpler mass matrices for charged particles.

3.1. CHARGED PARTICLES MASSES AND MIXING ANGLES

All of these masses have $\Delta I_{\mathrm{w}} = \frac{1}{2}$ because the left-handed particles are $\mathrm{SU}(2)_{\mathrm{L}}$ doublets. This follows from the nature of Dirac masses,

$$\text{Dirac mass} \sim \psi_{\mathrm{L}}^{+} \psi_{\mathrm{R}} + \text{c.c.}\,, \tag{3.1}$$

where ψ_{L} and ψ_{R} are two-component complex Weyl spinors. In order to keep track of the various contributions to a given sector of the mass matrices it is convenient to recall that, in SU(5) language,

$\langle \cdots \rangle \sim \underset{\sim}{5}$ contributes to charge $\frac{2}{3}$ masses and neutral Dirac masses with equal weight;

$\langle \cdots \rangle \sim \bar{\underset{\sim}{5}}$ contributes to charge $-\frac{1}{3}$ and -1 masses with equal weight;

$\langle \cdots \rangle \sim \underset{\sim}{45}$ contributes to charge $-\frac{1}{3}$ and -1 masses with a relative weight of -3 in the lepton sector.

In the case of the $\underset{\sim}{45}$ the relative factor of -3 can be easily understood because the allowable vev is in the $\underset{\sim}{126}$. Using the decomposition (2.18), we see that it resides in $(\underset{\sim}{2}, \underset{\sim}{2}, \underset{\sim}{15})$. The -3 factor comes simply from the relative value of the color singlet I_{15} for leptons and quarks. On the other hand, $\underset{\sim}{5}$ or $\bar{\underset{\sim}{5}}$ breaking resides in the $(\underset{\sim}{2}, \underset{\sim}{2}, \underset{\sim}{1})$ and therefore cannot distinguish colored from uncolored particles. Thus if we set for the $\Delta I_{\mathrm{w}} = \frac{1}{2}$ vev's

$$\langle 10_1 + i 10_2 \rangle = r e^{i\theta} (\text{along } \bar{\underset{\sim}{5}}) + p e^{i\delta} (\text{along } \underset{\sim}{5})\,, \tag{3.2}$$

$$\langle 126_1 \rangle = t e^{-i\sigma} (\text{along } \bar{\underset{\sim}{5}})\,, \tag{3.3}$$

$$\langle 126_2 \rangle = s e^{-i\chi} (\text{along } \underset{\sim}{45})\,, \tag{3.4}$$

$$\langle 126_3 \rangle = q e^{-i\mu} (\text{along } \bar{\underset{\sim}{5}})\,, \tag{3.5}$$

we will obtain the following mass matrices:

$$M_{-1/3}=\begin{pmatrix} 0 & Re^{i\theta} & 0 \\ Re^{i\theta} & Se^{i\chi} & 0 \\ 0 & 0 & Te^{i\theta} \end{pmatrix}, \qquad \text{for charge } -\tfrac{1}{3} \text{ quarks}, \tag{3.6}$$

$$M_{-1}=\begin{pmatrix} 0 & Re^{i\theta} & 0 \\ Re^{i\theta} & -3Se^{i\chi} & 0 \\ 0 & 0 & Te^{i\theta} \end{pmatrix}, \qquad \text{for charge } -1 \text{ leptons}, \tag{3.7}$$

$$M_{2/3}=\begin{pmatrix} 0 & Pe^{i\lambda} & 0 \\ Pe^{i\lambda} & 0 & Qe^{i\mu} \\ 0 & Qe^{i\mu} & Ve^{i\tau} \end{pmatrix}, \qquad \text{for charge } \tfrac{2}{3} \text{ quarks}. \tag{3.8}$$

In the above we have set

$$Pe^{i\lambda}=ape^{i\delta}+Ate^{i\sigma}, \tag{3.9}$$

$$Ve^{i\tau}=bpe^{i\delta}+Bte^{i\sigma}, \tag{3.10}$$

as well as

$$R=ar, \qquad T=br, \qquad S=cs, \qquad Q=dq. \tag{3.11}$$

Note that the presence of $\langle 126\rangle_1$ along $\underset{\sim}{\bar{5}}$ prevents us from being able to derive a wrong formula for the t-quark mass. At the same time, it makes the phase of the quark mass matrix essentially untractable; hence our model does not provide any answer to the strong CP problem.

We now proceed to diagonalize these mass matrices. It is convenient to first absorb the phases in the definition of the left- and right-handed field. To this effect we note that

$$M_{-1/3}=e^{i\Phi_L}\begin{pmatrix} 0 & R & 0 \\ R & S & 0 \\ 0 & 0 & T \end{pmatrix}e^{i\Phi_R}, \tag{3.12}$$

$$M_{-1}=e^{i\Phi_L}\begin{pmatrix} 0 & R & 0 \\ R & -3S & 0 \\ 0 & 0 & T \end{pmatrix}e^{i\Phi_R}, \tag{3.13}$$

$$M_{2/3}=e^{i\Delta_L}\begin{pmatrix} 0 & P & 0 \\ P & 0 & Q \\ 0 & Q & V \end{pmatrix}e^{i\Delta_R}, \tag{3.14}$$

where Φ_L, Φ_R, Δ_L, Δ_R are diagonal 3×3 matrices whose values are dictated by the phases appearing in (3.6)–(3.8). The relevant terms in the weak interaction lagrangian for the study of the diagonalization of these mass matrices are

$$\mathcal{L} = W^{\mu}\left[U_L^{+}\sigma_{\mu}D_L + N_L^{+}\sigma_{\mu}E_L\right] + U_L^{+}M_{2/3}U_R + D_L^{+}M_{-1/3}D_R + E_L^{+}M_{-1}E_R + \cdots , \tag{3.15}$$

where $U_L(U_R)$, $D_L(D_R)$, $E_L(E_R)$ are 3-component column vectors with entries of left (right)-handed charge $\frac{2}{3}, -\frac{1}{3}$, and -1 fields, respectively, which do not yet correspond to physical eigenstates. N_L is a neutral lepton 3-component vector to be elaborated on later. Using the decompositions (3.12)–(3.14), and absorbing the phases in the fields, we find

$$\begin{aligned}\mathcal{L} = {} & W^{\mu}\left[U_L^{+}\sigma_{\mu}e^{i(\Phi_L-\Delta_L)}D_L + N_L^{+}\sigma_{\mu}e^{i\Phi_L}E_L\right] + U_L^{+}\overline{M}_{2/3}U_R \\ & + D_L^{+}\overline{M}_{-1/3}D_R + E_L^{+}\overline{M}_{-1}E_R + \cdots ,\end{aligned} \tag{3.16}$$

where the bar over the mass matrices indicates that they have no phases left. The fields in the above have also been redefined. Thus the only effect of phases in the quark sector is the phase matrix in the charged current. This phase matrix can be written as

$$e^{i(\Phi_L-\Delta_L)} = \begin{pmatrix} e^{i\phi_1} & & \\ & e^{i\phi_2} & \\ & & e^{i\phi_3}\end{pmatrix},$$

with

$$\phi_1 = 2\theta-\chi-\lambda+a'-a, \qquad \phi_2=\theta+\tau-2\mu+a'-a, \qquad \phi_3=\theta-\mu+a'-c, \tag{3.17}$$

where a, a', c are arbitrary phases appearing in Φ_R and Δ_R. However, note that the first two entries cannot be simultaneously set equal to zero since the same combination of parameters enters in both. Hence our model will necessarily show CP-violation in the charged current sector. The diagonalization of the real matrices proceeds in a standard way. Set

$$\overline{M}_{-1/3} = R^{T}_{-1/3}\begin{pmatrix} -m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b\end{pmatrix}R_{-1/3}, \tag{3.18}$$

where R is the rotation matrix

$$R_{-1/3}=\begin{pmatrix}\cos\theta_c & -\sin\theta_c & 0\\ \sin\theta_c & \cos\theta_c & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{3.19}$$

and

$$\tan\theta_c=\sqrt{m_d/m_s}\,. \tag{3.20}$$

Next we write

$$\overline{M}_{-1}=R^{T}_{-1}\begin{pmatrix}m_e & 0 & 0\\ 0 & -m_\mu & 0\\ 0 & 0 & m_\tau\end{pmatrix}R_{-1}, \tag{3.21}$$

with

$$R_{-1}=\begin{pmatrix}\cos\beta & \sin\beta & 0\\ -\sin\beta & \cos\beta & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\tan\beta=\sqrt{m_e/m_\mu}\,. \tag{3.22}$$

Having identified the eigenstates of the mass matrices with the physical masses, it is easy to deduce by comparing (3.6) and (3.7) that

$$m_b=m_\tau, \tag{3.23}$$

and the Georgi-Jarlskog relations

$$m_d m_s=m_e m_\mu, \tag{3.24}$$

$$(m_d-m_s)=3(m_e-m_\mu), \tag{3.25}$$

which lead to

$$m_e/m_\mu=\tfrac{1}{9}m_d/m_s. \tag{3.26}$$

These relations are true at unification, but one does not expect much change in

(3.26) since it involves a ratio of quark masses. Finally

$$\overline{M}_{2/3} = R^{\mathrm{T}}_{2/3} \begin{pmatrix} m_{\mathrm{u}} & 0 & 0 \\ 0 & -m_{\mathrm{c}} & 0 \\ 0 & 0 & m_{\mathrm{t}} \end{pmatrix} R_{2/3}, \tag{3.27}$$

where the rotation matrix $R_{2/3}$ is more complicated. However, to $\mathrm{O}(m_{\mathrm{u}}/m_{\mathrm{c}}, m_{\mathrm{u}}/m_{\mathrm{t}})$, it is given by

$$R_{2/3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{3.28}$$

with

$$\tan\alpha = \sqrt{m_{\mathrm{c}}/m_{\mathrm{t}}}\,.$$

We now introduce the physical (mass eigenstate) fields

$$E^{\mathrm{phys}}_{\mathrm{L}} = R_{-1}E_{\mathrm{L}}, \qquad D^{\mathrm{phys}}_{\mathrm{L}} = R_{-1/3}D_{\mathrm{L}}, \qquad U^{\mathrm{phys}}_{\mathrm{L}} = R_{2/3}U_{\mathrm{L}}. \tag{3.29}$$

The charged current now looks like

$$j_{\mu} = N^{+}_{\mathrm{L}}\sigma_{\mu}e^{i\Phi_{\mathrm{L}}}R^{\mathrm{T}}_{-1}E^{\mathrm{phys}}_{\mathrm{L}} + U^{+\mathrm{phys}}_{\mathrm{L}}\sigma_{\mu}R_{2/3}e^{i(\Phi_{\mathrm{L}}-\Delta_{\mathrm{L}})}R_{-1/3}D^{\mathrm{phys}}_{\mathrm{L}}. \tag{3.30}$$

The mixing matrix for the quark charged current can be put in the standard KM form by identifying $\theta_1 = \theta, \theta_2 = \alpha, \theta_3 = 1, \delta_{\mathrm{KM}} = 2(\mu - \delta) - \nu - \chi$. Thus our model has CP violation and the interesting thing is that the strength of the $\mathrm{b} \to \mathrm{c}$ transition directly measures the $m_{\mathrm{c}}/m_{\mathrm{t}}$ mass ratio. In fact, our matrix gives the lifetime of the B-meson to be

$$\tau_{\mathrm{B}} = \left(\frac{m_{\mathrm{t}}}{m_{\mathrm{c}}}\right) 4.45 \times 10^{-15}\,\mathrm{sec}, \tag{3.31}$$

to be compared with the existing limit

$$\tau_{\mathrm{B}} \lesssim 10^{-11}\,\mathrm{sec}, \tag{3.32}$$

which provides a weak upper bound for m_{t}.

Let us summarize the results of this section. To $\mathrm{O}(m_{\mathrm{u}}/m_{\mathrm{c}}, m_{\mathrm{u}}/m_{\mathrm{t}})$ we have found

$$\tan\theta_{\mathrm{c}} = \sqrt{m_{\mathrm{d}}/m_{\mathrm{s}}}\,, \qquad \tan\alpha = \sqrt{m_{\mathrm{c}}/m_{\mathrm{t}}}\,, \tag{3.33}$$

$$m_{\mathrm{b}} = m_{\tau}, \qquad m_{\mathrm{e}}/m_{\mu} = \tfrac{1}{9}m_{\mathrm{d}}/m_{\mathrm{s}}, \tag{3.34}$$

as well as (3.31), all such relations occurring in a natural field-theoretical way. Unfortunately, it does not predict m_t except through τ_B and does not address itself to the strong CP problem.

3.2. NEUTRAL PARTICLE MASSES AND MIXING ANGLES

The mass matrices of neutral colorless particles may be of the lepton-number violating Majorana or of the lepton-number conserving Dirac type. Thus it is important to preface this section with a general discussion of the various masses allowed by Lorentz invariance.

The Lorentz group is algebraically equivalent to SU(2) × SU(2), which leads to the classification of its representations by two numbers (S_1, S_2), where S_1 and S_2 denote the spin of each SU(2). However, since the Lorentz group is non-compact, these two SU(2)'s are not unitarily realized so that they can be related to one another by conjugation as well as by parity flip. Indeed if $\boldsymbol{J}$ and $\boldsymbol{K}$ are the rotation and boost hermitian generators, respectively, $\boldsymbol{J}+i\boldsymbol{K}$ and $\boldsymbol{J}-i\boldsymbol{K}$ generate the two SU(2)'s. The simplest non-trivial representations are realized by complex two-component spinors (Weyl spinors). They are

$$\psi_L \sim (\tfrac{1}{2},0), \qquad \psi_R \sim (0,\tfrac{1}{2}). \tag{3.35}$$

Physically $\psi_{L(R)}$ can be identified with a left (right)-handed massless particle, L(R) denoting its helicity. Parity flips ψ_L into ψ_R and vice versa, so that a parity-conserving theory must involve both ψ_L and ψ_R. Under the Lorentz group,

$$\psi_{L(R)} \to e^{(i\sigma/2)\cdot(\omega \pm i\nu)}\psi_{L(R)}, \tag{3.36}$$

where $\omega(\nu)$ are the rotation (boost) parameters. It follows then that the new field $\sigma_2\psi_L^*$ transforms as ψ_R and $\sigma_2\psi_R^*$ as ψ_L:

$$\sigma_2\psi_L^* \sim (0,\tfrac{1}{2}), \qquad \sigma_2\psi_R^* \sim (\tfrac{1}{2},0). \tag{3.37}$$

We therefore identify the transformations

$$\psi_L \to \sigma_2\psi_R^*, \qquad \psi_R \to -\sigma_2\psi_L^*, \tag{3.38}$$

with the operation of charge conjugation which changes a left (right)-handed particle into a left (right)-handed antiparticle. Thus we can identify CP with

$$CP: \quad \psi_L \to -\sigma_2\psi_L^*, \qquad \psi_R \to \sigma_2\psi_R^*, \tag{3.39}$$

since it changes a left-handed particle into a right-handed antiparticle. As an example, ψ_L can represent a left-handed neutrino, and $\sigma_2\psi_L^*$ a right-handed antineutrino.

It is clear that any theory involving spin $\frac{1}{2}$ fields can be written in terms of purely left-handed Weyl fields. With such spinors one can construct two types of masses. Starting with ψ_L alone, one can construct the Majorana mass, corresponding to

$$[(\tfrac{1}{2},0)\times(\tfrac{1}{2},0)]_A=(\underset{\sim}{0},\underset{\sim}{0}), \tag{3.40}$$

and it is of the form

$$\psi_L^T\sigma_2\psi_L \qquad [\text{Majorana mass}]. \tag{3.41}$$

When more than one ψ_L is present one can form another type of mass, the Dirac mass; it involves two types of Weyl spinors. Traditionally it is written in the form $\psi_L^+\psi_R+$ c.c., but since it is always possible to interpret ψ_R as $\sigma_2\psi_L'^*$ we can regard it as an off-diagonal Majorana mass. For instance in a theory with two Weyl spinors ψ_{L1} and ψ_{L2}, the most general mass matrix is

$$i\psi_{La}^T M_{ab}\sigma_2\psi_{Lb}+\text{c.c.}, \tag{3.42}$$

where M is a symmetric 2×2 matrix. Its off-diagonal elements give rise to Dirac masses through the identification $\psi_{L2}=\sigma_2\psi_{R1}^*$, while its diagonal elements are of the true Majorana type.

In our case we have 2 neutral Weyl spinors per family, that is a total of 6 neutral lepton spinors. Thus we have to diagonalize a mass term of the form (3.42) when $a,b=1,\dots,6$. Of these six entries three of them are $SU(2)_w$ doublets and three are singlets. Hence the structure of the 6×6 matrix to be diagonalized is

$$M_0=\begin{pmatrix} M^{(0)} & M^{(1/2)} \\ M^{(1/2)T} & M^{(1)} \end{pmatrix}, \tag{3.43}$$

where the $M^{(\Delta I_w)}$ are 3×3 matrices, the superscript standing for their ΔI_w value. Since M is symmetric, $M^{(0)}$ and $M^{(1)}$ are symmetric matrices. In principle the 126's can assume $\Delta I_w=1$ vacuum values through the 15 of SU(5), thus contributing to $M^{(1)}$. However, we neglect such contributions by taking them to be much smaller than the $\Delta I_w=\frac{1}{2}$ entries squared, divided by the $\Delta I_w=0$ entries. If $M^{(0)}$ is generated by radiative corrections as suggested in [7], then this will always be the case [8]. If $M^{(0)}$ is generated by explicit $\Delta I_w=0$ vacuum expectation values as in this model, then this assumption depends on the detailed form of the Higgs potential and is not technically natural for the most general potential. In what follows we will nevertheless take

$$M^{(1)}=0. \tag{3.44}$$

In SO(10), the $\Delta I_w=\frac{1}{2}$ mass matrices of the charge $\frac{2}{3}$ and 0 sectors are related by

Clebsch-Gordan coefficients: the $\underset{\sim}{10}$, having its vev along $(\underset{\sim}{2},\underset{\sim}{2},\underset{\sim}{1})$ in the chiral decomposition gives equal weight to leptons and quarks, while the 126 having its $\Delta I_w = \frac{1}{2}$ vev along $(\underset{\sim}{2},\underset{\sim}{2},\underset{\sim}{15})$ gives leptons a factor of -3 relative to the quarks. Hence we can identify

$$M^{(1/2)} = \begin{pmatrix} 0 & Pe^{i\lambda} & 0 \\ Pe^{i\lambda} & 0 & -3Qe^{i\mu} \\ 0 & -3Qe^{i\mu} & Ve^{i\tau} \end{pmatrix}, \tag{3.45}$$

by comparison with the matrix (3.8).

In terms of the light quark masses,

$$P = \left\{ \frac{m_u m_c m_t}{m_t - m_c + m_u} \right\}^{1/2}, \qquad V = m_t - m_c + m_u,$$

$$Q = \left\{ \frac{(m_t - m_c)(m_c - m_u)(m_t - m_u)}{m_t - m_c + m_u} \right\}^{1/2}. \tag{3.46}$$

The form of $M^{(0)}$ is dictated solely by the vev of 126_1 along the $\Delta I_w = 0$ direction. We have

$$M^{(0)} = \begin{pmatrix} 0 & \bar{A}e^{i\zeta} & 0 \\ \bar{A}e^{i\zeta} & 0 & 0 \\ 0 & 0 & \bar{B}e^{i\zeta} \end{pmatrix}, \tag{3.47}$$

where

$$\bar{A} = Ak, \qquad \bar{B} = Bk, \tag{3.48}$$

and the $\Delta I_w = 0$ vev of 126_1 is

$$\langle 126_1 \rangle = ke^{i\zeta}(\text{along } \underset{\sim}{1}). \tag{3.49}$$

Thus the 6×6 neutral lepton mass matrix is symmetric and complex, allowing for CP violation. Its diagonalization proceeds by means of Schur's theorem [9] which says that a complex symmetric matrix can be diagonalized by means of a unitary congruence; that is, we can write

$$M_0 = U^T D U, \tag{3.50}$$

where D is a diagonal matrix and U is a unitary matrix

$$U^+ U = UU^+ = 1. \tag{3.51}$$

The entries of D can be obtained by taking the square root of the eigenvalues of the hermitian matrix M^*M since

$$M^*M = U^+ D^2 U. \tag{3.52}$$

The formidable mathematical task of handling a 6×6 matrix is somewhat alleviated by the fact that the entries in $M^{(1/2)}$ are much smaller than the entries in $M^{(0)}$, because of the gauge hierarchy. In fact, let us set

$$M^{(1/2)} = \varepsilon \hat{M}^{(1/2)}, \tag{3.53}$$

where $\hat{M}^{(1/2)}$ is of the same order of magnitude as $M^{(0)}$, and ε measures the relative strength of the $\Delta I_w = \frac{1}{2}$ to $\Delta I_w = 0$ breakings

$$\varepsilon = \frac{\Delta I_w = \frac{1}{2}}{\Delta I_w = 0} \ll 1. \tag{3.54}$$

Then we set the unitary matrix

$$U = \begin{pmatrix} U_{11} & \varepsilon U_{12} \\ \varepsilon U_{21} & U_{22} \end{pmatrix}, \tag{3.55}$$

where the 3×3 matrices U_{11}, U_{22}, U_{12} and U_{21} are of the same order, and

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & \varepsilon^2 D_2 \end{pmatrix}, \tag{3.56}$$

where D_1 and D_2 are 3×3 diagonal matrices of the same order. Then as a consequence of the unitarity of U and of the unitary congruence (3.50) we obtain the set of 3×3 matrix equations

$$M^{(0)} = U_{11}^{\mathrm{T}} D_1 U_{11} + \mathrm{O}(\varepsilon^2), \tag{3.57}$$

$$\hat{M}^{(1/2)\mathrm{T}} \frac{1}{M^{(0)}} \hat{M}^{(1/2)} = -U_{22}^{\mathrm{T}} D_2 U_{22}, \tag{3.58}$$

U_{11} and U_{22} being unitary matrices to $\mathrm{O}(\varepsilon^2)$. This last equation is the matrix equivalent of the G-MRS mechanism. From (3.57) we solve for U_{11} and D_1. Next we solve the 3×3 congruence problem (3.58) to find U_{22} and D_2.

Clearly D_1 gives the masses for the heavy neutrinos and $\varepsilon^2 D_2$ the masses of the light neutrinos of the model. Furthermore, in terms of the physical mass eigenstates it is easy to see that the lepton charged current is given by

$$j_\mu = \nu_{\mathrm{L}}^+ \sigma_\mu U_{22} e^{i\Phi_{\mathrm{L}}} R_{-1}^{\mathrm{T}} E_{\mathrm{L}}^{\mathrm{phys}} + \mathrm{O}(\varepsilon), \tag{3.59}$$

where ν_{L} is a 3-component vector with entries corresponding to the light neutrinos.

From (3.57) we see that

$$D_1 = \begin{pmatrix} \bar{A} & 0 & 0 \\ 0 & -\bar{A} & 0 \\ 0 & 0 & \bar{B} \end{pmatrix}, \qquad U_{11} = \sqrt{\tfrac{1}{2}}\, e^{i\zeta/2} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \tag{3.60}$$

so that we have two heavy right-handed neutrinos of mass $M_1 = \bar{A}$ and one of mass $M_2 = \bar{B}$. (The negative sign in front of the second eigenvalue can be absorbed by a chiral transformation on N_μ, the heavy neutrino eigenstate associated with the second family.)

We now consider the diagonalization of the light neutrino mass matrix,

$$M_\nu = M^{(1/2)\mathrm{T}} M^{(0)-1} M^{(1/2)} = -U_{22}^{\mathrm{T}} \varepsilon^2 D_2 U_{22}. \tag{3.61}$$

By taking the determinant of M_ν we find that the neutrino masses must obey the constraint

$$m_{\nu_e} m_{\nu_\mu} m_{\nu_\tau} = \frac{m_u^2 m_c^2 m_t^2}{m_{N_e} m_{N_\mu} m_{N_\tau}}, \tag{3.62}$$

where the overall phase has been absorbed in U_{22}. Using (3.45) and (3.47) we find that all but one of the phases appearing in M_ν may be absorbed in redefinition of the light neutrino fields:

$$M_\nu = e^{i\Phi_\nu} \begin{pmatrix} 0 & P^2/\bar{A} & 0 \\ P^2/\bar{A} & (9Q^2/\bar{B})e^{i\mu} & -3Q(P/\bar{A} + V/\bar{B}) \\ 0 & -3Q(P/\bar{A} + V/\bar{B}) & V^2/\bar{B} \end{pmatrix} e^{i\Phi_\nu} \tag{3.63}$$

where Φ_ν is a 3×3 real diagonal matrix. The remaining phase appearing in the 2-2 entry cannot be removed without complete diagonalization of M_ν. For simplicity we will take $\mu = 0$. In general, the neutrino masses and mixing angles will be functions of μ. For $\mu = 0, U_{22}$ is an orthogonal matrix to $O(\varepsilon^2)$ which we parametrize as

$$U_{22} = \begin{pmatrix} c_1 & s_1 c_2 & s_1 s_2 \\ -s_1 c_3 & c_1 c_2 c_3 - s_2 s_3 & c_1 s_2 c_3 + c_2 s_3 \\ -s_1 s_3 & c_1 c_2 s_3 + s_2 c_3 & c_1 s_2 s_3 - c_2 c_3 \end{pmatrix}, \tag{3.64}$$

with $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $i = 1,2,3$.

An analytic solution for U_{22} and D_2 is complicated and not very enlightening. Instead, we will analyze the eigenvalues and mixing angles for two limiting cases that illustrate the general features and then present the results of a numerical solution.

It is convenient to consider the neutrino masses to depend on m_u, m_c, and m_t through P, Q, and V, and on the ratio $r = m_{N_e}/m_{N_\tau} = \bar{A}/\bar{B}$ with the overall scale set by $\bar{A}$. We first suppose that the right-handed neutrino masses obey a family hierarchy similar to that for the observed leptons and quarks so that $r \sim m_u/m_t \ll 1$. In the approximation that

$$V \simeq m_t, \qquad P \simeq \sqrt{m_u m_c}\,, \qquad Q \simeq \sqrt{m_t m_c}\,, \tag{3.65}$$

we find that

$$m_{\nu_e} \simeq r m_t m_u / 9\bar{A}, \qquad m_{\nu_\mu} \simeq m_{\nu_\tau} \simeq 3 m_c \sqrt{m_t m_u}\,/\bar{A}, \tag{3.66}$$

with the mixing angles given approximately by

$$\theta_1 \simeq \tfrac{1}{3}\sqrt{m_u/m_t}\,, \qquad \theta_2 \simeq \tfrac{1}{2}\pi - \frac{r m_t}{3 m_c}\sqrt{m_t/m_u}\,, \qquad \theta_3 \simeq \tfrac{1}{4}\pi, \tag{3.67}$$

so that appreciable mixing exists only in the μ-τ sector*. If m_{ν_μ} and m_{ν_τ} saturate the cosmological bound $\Sigma m_\nu \lesssim 100$ eV [11], then we find $m_{\nu_e} \sim 3.75r$ eV for $m_t = 25$ GeV.

If the right-handed neutrino masses are all equal so that $r = 1$ we then find to the same approximation that

$$m_{\nu_e} \simeq \frac{\left(m_c m_u^3\right)^{1/2}}{18\bar{A}}, \qquad m_{\nu_\mu} \simeq \frac{18 m_t}{\bar{A}(m_t + 9m_c)}\left(m_c^3 m_u\right)^{1/2},$$

$$m_{\nu_\tau} \simeq \frac{m_t(m_t + 9m_c)}{\bar{A}}, \tag{3.68}$$

with the mixing angles given by

$$\theta_1 \simeq \frac{1}{18}\left[\frac{m_u(m_t + 9m_c)}{m_t m_c}\right]^{1/2}, \qquad \theta_2 \simeq \cos^{-1}\left[\frac{m_t}{m_t + 9m_c}\right], \qquad \theta_3 \simeq \pi, \tag{3.69}$$

so that again appreciable mixing exists only between μ and τ neutrinos. If m_{ν_τ} saturates the cosmological bound then for $m_t = 25$ GeV we find $m_{\nu_e} \simeq 1 \times 10^{-3}$ eV. Note that for $r = 1$, m_{ν_μ} differs from the case $r \ll 1$ by a factor of $6\sqrt{m_t m_c}\,/(m_t + 9m_c)$ which is independent of r and varies from 0.98 to 0.82 for m_t between 20 and 50 GeV. We find for all values of r that m_{ν_μ} is essentially constant and therefore provides the best measurement of the mass scale for the right-handed neutrinos. Numerical results for the masses and mixing angles are given in table 1 for r varying

* The issue of the value of neutrino mixing angles has been widely discussed. See, for example, [10].

TABLE 1

Masses and mixing angles for light neutrinos as a function of $r = m_{N_e}/m_{N_\tau}$ and top quark mass m_t

m_t (GeV)	$\log_{10} r$	$\bar{a}m_{\nu_e}$ (ev)	$\bar{a}m_{\nu_\mu}$ (ev)	$\bar{a}m_{\nu_\tau}$ (ev)	θ_1 (rad)	θ_2 (rad)	θ_3 (rad)
20	−4	9.91×10^{-8}	0.148	0.154	0.0054	1.548	2.378
20	−2	2.02×10^{-6}	0.147	0.759	0.0041	0.9217	2.965
20	0	2.50×10^{-6}	0.147	61.3	0.0041	0.7277	3.138
25	−4	1.24×10^{-7}	0.164	0.173	0.0048	1.538	2.386
25	−2	2.09×10^{-6}	0.160	1.05	0.0039	0.8223	3.015
25	0	2.48×10^{-6}	0.159	89.0	0.0039	0.6644	3.136
30	−4	1.49×10^{-7}	0.178	0.190	0.0043	1.527	2.395
30	−2	2.14×10^{-6}	0.170	1.39	0.0037	0.7462	3.048
30	0	2.47×10^{-6}	0.168	122	0.0037	0.6154	3.134

m_{N_i} is the mass of the ith supermassive right-handed neutrino field. m_{ν_i} is the mass in electron volts of the ith neutrino. $\bar{a} = m_{N_e}/10^{10}$ GeV. The mixing angles are given in radians.

from 10^{-4} to 1 and for $m_t = 20$, 25 and 30 GeV. We find that the previous approximations are good to about 10% when applicable.

Neutrino oscillations are governed by the mixing matrix

$$K = U_{22} e^{i\Phi_L} R_{-1}^{T}, \tag{3.70}$$

which appears in the leptonic charged current (see e.g. [12]). The probability of finding a weak eigenstate neutrino ν_ℓ (i.e. the neutrino that would accompany the lepton ℓ in a weak decay) at a time t in a beam of momentum p which consisted of only $\nu_{\ell'}$ at $t = 0$ is given by

$$P_{\ell'\ell}(t) = \sum_{i,j} K_{\ell' i} K_{\ell i}^{*} K_{\ell' j}^{*} K_{\ell j} e^{-i(E_i - E_j)t}. \tag{3.71}$$

In the relativistic limit the relation $(E_i - E_j)t = 2\pi R/L$ holds, where R is the distance from the source of the beam and

$$L = \frac{4\pi p}{|m_{\nu_i}^2 - m_{\nu_j}^2|} \tag{3.72}$$

is the oscillation length.

In principle, the CP-violating phases appearing in (3.71) are measurable since they affect the cosine dependence of the probability for neutrino oscillations [13]. In practice, such phases will be very difficult to measure and we will ignore them in what follows. To a good approximation we can set $\theta_1 = 0$ so that

$$K = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta\cos\theta & \cos\beta\cos\theta & \sin\theta \\ \sin\beta\sin\theta & \cos\beta\sin\theta & -\cos\theta \end{pmatrix}, \tag{3.73}$$

TABLE 2
Parameters for ν_μ-ν_τ oscillations

m_t (GeV)	$\log_{10} r$	$\bar{a}^2 \delta m^2$ (ev)	$\sin^2 2\theta$
20	−4	0.002	0.999
20	−2	0.554	0.994
20	0	3758	0.984
25	−4	0.003	0.999
25	−2	1.077	0.968
25	0	7921	0.937
30	−4	0.004	0.999
30	−2	1.903	0.931
30	0	14884	0.879

δm^2 is the difference of the squared masses of ν_μ and ν_τ. θ is the mixing angle connecting weak and mass eigenstates.

with $\theta = \theta_2 + \theta_3$ and $\tan\beta = \sqrt{m_e/m_\mu}$ as before. In this model the neutrino oscillations are thus parametrized by the three squared mass differences of the neutrinos and by the mixing angles θ and β. Since $m_{\nu_e} \ll m_{\nu_\mu}, m_{\nu_\tau}$, we give in table 2 only the difference $m^2_{\nu_\tau} - m^2_{\nu_\mu}$ and the value of θ for the same range of r and m_t as in table 1.

Our model shows close to maximal mixing between ν_μ and ν_τ since $\tan\beta \ll 1$. Recent results on the lack of observation of ν_μ-ν_τ oscillations indicate that $m^2_{\nu_\tau} - m^2_{\nu_\mu} < 3$ ev^2 for maximum mixing. This allows us to derive bounds on the mass of the heavy neutral leptons, as well as the usual neutrinos. The results are summarized in table 3.

Finally, let us emphasize that our model gives a tiny electron neutrino mass and very small ν_e oscillations. The results obtained here depend to some degree on the simple form of the $\Delta I_w = 0$ mass matrix. Still the general features such as the

TABLE 3
Bounds on neutrino masses from Brookhaven-Columbia E53A, as applied to our model

m_t (GeV)	$\log_{10} r$	m_{N_e} > (GeV)	m_{ν_e} < (ev)	m_{ν_μ} < (ev)	m_{ν_τ} < (ev)
20	−4	0.026×10^{10}	3.81×10^{-6}	5.69	5.92
20	−2	0.429×10^{10}	4.71×10^{-6}	0.343	1.77
20	0	35.7×10^{10}	7.00×10^{-8}	0.004	1.72
25	−4	0.032×10^{10}	3.88×10^{-6}	5.13	5.41
25	−2	0.599×10^{10}	3.49×10^{-6}	0.267	1.75
25	0	52.6×10^{10}	4.71×10^{-8}	0.003	1.69
30	−4	0.037×10^{10}	4.03×10^{-6}	4.81	5.14
30	−2	0.794×10^{10}	2.70×10^{-6}	0.214	1.75
30	0	71.4×10^{10}	3.46×10^{-8}	0.002	1.71

smallness of the ν_e mass and the predominance of ν_μ-ν_τ oscillations are tied to the dependence of the neutrino masses and mixings on the charge $\frac{2}{3}$ quark mass matrix, which is a general feature of all SO(10) grand unified theories.

4. The Higgs potential

The Achilles' heel of all grand unified theories is the specification of the mechanism which breaks the symmetries and gives the fermions their masses. In this paper we rely on explicit Higgs (spinless) fields to perform these functions because they give the most efficient renormalizable parametrization of this mechanism. Simplicity of the theory in the vector and spinors sectors seems to invariably result in a complicated Higgs structure ("law of conservation of trouble"). Our theory is no exception to this rule, but we feel it instructive to go with the explicit Higgs even though we know it to be too complicated to be true. It is likely that a large number of the Higgs fields will be understood either as effective two-fermion or effective two-Higgs fields.

In theories where the Higgs fields are composite or result from higher order corrections the questions of the naturalness of the assumed breaking and the forms of the allowed couplings must still be faced. The techniques used in the following analysis may be used to address these questions independently of the nature of the Higgs fields.

In our model, the Higgs potential consists of all the SO(10)-invariant quadratic, cubic and quartic interactions among the $\underset{\sim}{10}$, $\underset{\sim}{54}$, $\underset{\sim}{126}_1$, $\underset{\sim}{126}_2$ and $\underset{\sim}{126}_3$ spinless fields, and which preserve the global X and Y symmetries of the Yukawa terms mod 4 and mod 8, respectively. For convenience, we split up the potential into terms that involve only one type of field and terms with several Higgs.

The simplest is the one involving only the $\underset{\sim}{54}$. From the products

$$(\underset{\sim}{54}\times\underset{\sim}{54})_S=\underset{\sim}{1}+\underset{\sim}{54}+\underset{\sim}{660}+\underset{\sim}{770}\,, \tag{4.1}$$

$$(\underset{\sim}{54}\times\underset{\sim}{54}\times\underset{\sim}{54})_S=\underset{\sim}{1}+\underset{\sim}{54}+\underset{\sim}{54}+\cdots\,, \tag{4.2}$$

where the dots stand for representations other than $\underset{\sim}{1}$ or $\underset{\sim}{54}$, we know that the most general form is

$$V_{54}=\left(54\cdot 54;\ 54\cdot 54\cdot 54;\ (54\cdot 54)_1^2,\ (54\cdot 54)_{54}^2\right). \tag{4.3}$$

Here $(54\cdot 54)_{54}^2$ denotes the square of the projection of $54\cdot 54$ onto the 54. The cubic term can be neglected in a natural way since its absence is guaranteed by the discrete symmetry $54\to -54$.

The form of V_{10} is dictated, beside SO(10) invariance, by the (X, Y) value of the (complexified) 10 of (1,0). We use the products

$$\underset{\sim}{10} \times \underset{\sim}{10} = (\underset{\sim}{1} + \underset{\sim}{54})_S + (\underset{\sim}{45})_A, \tag{4.4}$$

$$\underset{\sim}{10} \times \underset{\sim}{10} \times \underset{\sim}{10} = (\underset{\sim}{120})_A + (\underset{\sim}{210} + \underset{\sim}{10})_S + (\underset{\sim}{320} + \underset{\sim}{320} + \underset{\sim}{10} + \underset{\sim}{10})_M, \tag{4.5}$$

where the subscript M denotes mixed symmetry. Since we have two real $\underset{\sim}{10}$'s (or one complexified $\underset{\sim}{10}$) we can have at most two quadratic invariants and six quartic invariants. However, by imposing X mod 4, we reduce their number to one quadratic invariant $(10 \cdot \overline{10})$ and four quartic invariants. These are of the form

$$V_{10} = \left\{ (10 \cdot \overline{10})_1;\ (10 \cdot \overline{10})_1^2,\ (10 \cdot \overline{10})_{54}^2,\ (10 \cdot 10)_1^2,\ (10 \cdot 10)_{54}^2 \right\}. \tag{4.6}$$

It is easy to see that the other possible quartic invariant $(10 \cdot \overline{10})_{45}^2$ can be expressed in terms of the other four. The last two quartic invariants respect X mod 4 and serve to avoid the massless Nambu-Goldstone boson that would ensue when the global X symmetry is spontaneously broken.

The terms in the potential that involve only one kind of 126 are more complicated. Let us for a moment neglect the (X, Y) symmetries and construct the most general potential containing only one type of $\underset{\sim}{126}$. Knowledge of the products

$$\underset{\sim}{126} \times \underset{\sim}{126} = (\underset{\sim}{54} + \underset{\sim}{1050} + \underset{\sim}{2772} + \underset{\sim}{4125})_S + (\underset{\sim}{945} + \underset{\sim}{6930})_A, \tag{4.7}$$

$$\underset{\sim}{126} \times \overline{\underset{\sim}{126}} = \underset{\sim}{1} + \underset{\sim}{45} + \underset{\sim}{210} + \underset{\sim}{770} + \underset{\sim}{5940} + \underset{\sim}{8910} \tag{4.8}$$

is extremely useful. Note that although the 126 is complex, only the $\underset{\sim}{1050}$, $\underset{\sim}{2772}$ and 6930 are complex. The number of independent quartic invariants is given by the number of times the $\overline{\underset{\sim}{126}}$ appears in the products $\underset{\sim}{126} \cdot \underset{\sim}{126} \cdot \underset{\sim}{126}$ and $\overline{\underset{\sim}{126}} \cdot \overline{\underset{\sim}{126}} \cdot \underset{\sim}{126}$. These are hard to calculate and we use powerful techniques to do so (see ref. [14]). The task is made easier when only one $\underset{\sim}{126}$ is present. We quote the results: the $\overline{\underset{\sim}{126}}$ appears once in the symmetric product $\underset{\sim}{126} \cdot \underset{\sim}{126} \cdot \underset{\sim}{126}$ and four times in $\overline{\underset{\sim}{126}} \cdot \overline{\underset{\sim}{126}} \cdot \underset{\sim}{126}$. Thus

$$\begin{aligned} V_{126} = \Big\{ &(126 \cdot \overline{126})_1;\ (126 \cdot \overline{126})_1^2,\ (126 \cdot 126)_{54} \cdot (\overline{126} \cdot \overline{126})_{54}, \\ &(126 \cdot 126)_{1050} \cdot (\overline{126} \cdot \overline{126})_{\overline{1050}},\ (126 \cdot 126)_{4125}\, (\overline{126} \cdot \overline{126})_{4125}, \\ &(126 \cdot 126)_{54}^2 \Big\}. \end{aligned} \tag{4.9}$$

There will be a set of such terms for each of the $\underset{\sim}{126}$'s. The 126_1 with $(X, Y) = (-1, 0)$

will have a $(126_1)^4$ term which breaks X mod 4 and preserves Y; the 126_2, with $(X, Y)=(1,-2)$ will exhibit a $(126_2)^4$ term which turns out to be the only term in the potential that breaks Y mod 8, thus avoiding a massless Nambu-Goldstone boson associated with the breaking of the global Y symmetry; the 126_3 with $(X, Y)=(0,-1)$ does not show a $(126_3)^4$ term and its absence is assured by the fact that Y is broken mod 8 (by the $(126)_2^4$ as discussed previously).

Now for terms involving two types of Higgs. First, from (4.1), (4.4) and (X, Y) conservation we easily see that

$$V_{10\text{-}54}=\left\{\left(10\cdot\overline{10}\right)_{54}\cdot 54;\ \left(10\cdot\overline{10}\right)_1\cdot(54\cdot 54)_1,\ \left(10\cdot\overline{10}\right)_{54}\cdot(54\cdot 54)_{54}\right\}. \tag{4.10}$$

Again the cubic term need not be present.

Second, from (4.1) and (4.8) it is easy to see that we have

$$V_{126\text{-}54}=\left\{\left(126\cdot\overline{126}\right)_1\cdot(54\cdot 54)_1,\ \left(126\cdot\overline{126}\right)_{770}\cdot(54\cdot 54)_{770}\right\} \tag{4.11}$$

for *each* $\underset{\sim}{126}$. Other possibilities such as $(126)^2_{54}\cdot(54)^2_{54}$ are ruled out by the discrete (X, Y) symmetries.

Third, using (4.4), (4.7) and (4.8) and

$$\underset{\sim}{10}\times\underset{\sim}{126}=\underset{\sim}{1050}+\underset{\sim}{210} \tag{4.12}$$

we have

$$V_{10\text{-}126_1}=\left\{\left(10\cdot\overline{10}\right)_1\cdot\left(126_1\cdot\overline{126_1}\right)_1,\ \left(10\cdot\overline{10}\right)_{45}\cdot\left(126_1\cdot\overline{126_1}\right)_{45},\right.$$
$$\left.(10\cdot 10)_{54}\cdot\left(\overline{126_1}\cdot\overline{126_1}\right)_{54}\right\}, \tag{4.13}$$

$$V_{10\text{-}126_2}=\left\{\left(10\cdot\overline{10}\right)_1\cdot\left(126_2\cdot\overline{126_2}\right)_1,\ \left(10\cdot\overline{10}\right)_{45}\cdot\left(126_2\cdot\overline{126_2}\right)_{45}\right\}, \tag{4.14}$$

$$V_{10\text{-}126_3}=\left\{\left(10\cdot\overline{10}\right)_1\cdot\left(126_3\cdot\overline{126_3}\right)_1,\ \left(10\cdot\overline{10}\right)_{45}\cdot\left(126_3\cdot\overline{126_3}\right)_{45}\right\}, \tag{4.15}$$

as well as quartic terms linear in the 10:

$$\left(\overline{10}\cdot 126_1\right)_{210}\cdot\left(126_i\cdot\overline{126_i}\right)_{210},\qquad i=1,2,3, \tag{4.16}$$

$$\left(10\cdot 126_3\right)_{210}\cdot\left(126_3\cdot\overline{126_2}\right)_{210}. \tag{4.17}$$

Finally we have terms that involve the different $\underset{\sim}{126}$'s; they are just of the form

$$\left(126_i\cdot 126_j\right)\cdot\left(\overline{126_i}\cdot\overline{126_j}\right),\qquad i\neq j=1,2,3.$$

There are three such types of terms corresponding to different values of (i, j) and from (4.7) and (4.8) we see that we have at most 6 different coupling schemes for each, that is 18 terms! Finally, there is a term of the form $(126_1 \cdot 126_2)\cdot(\overline{126}_3 \cdot \overline{126}_3)$. Thus our potential is made up of five quadratic invariants, one for each Higgs representation, and 59 (fifty nine!) independent quartic invariants, not counting those linear in the $\underset{\sim}{10}$ and the $\underset{\sim}{54}$. It is clear that no fundamental theory should have such a complicated structure. As we have previously stated, we consider the fundamental Higgs to represent a parametrization of the symmetry breaking.

Without going into the details of the potential, one can check if our required set of vacuum expectation values can be maintained in perturbation theory. The principle is this: expand any Higgs field about its desired vev and check that the magnitudes of the vev's can be adjusted so that the potential does not contain any terms linear in the (expanded) fields. This must be shown to be possible for a finite range of the parameters in the Higgs potential. If a particular linear term comes only from one invariant or if the above-mentioned constraints are incompatible with one another, the postulated set of vacuum expectation values is not natural. In our case it is convenient to expand the fields in SU(5) language: for instance set

$$10 = \langle 5\rangle_{10} + \langle\bar{5}\rangle_{10} + \zeta_5 + \zeta_{\bar{5}}, \tag{4.18}$$

$$126_1 = \langle 1\rangle_{126_1} + \langle\bar{5}\rangle_{126_1} + \chi_1 + \chi_{\bar{5}} + \chi_{10} + \chi_{\overline{15}} + \chi_{45} + \chi_{50}, \qquad \text{etc.}, \tag{4.19}$$

and check that it is possible to choose the vev so that terms linear in the ζ's, χ's, ... vanish. Now, this is a terrible task but it can be somewhat alleviated by enforcing a gauge hierarchy: some vev are much smaller than others. Call the small (large) vev's v (V). Then when we expand the potential and look at linear terms we demand that linear terms be made to vanish to each order in v and V. As an example, the terms of $O(vV^2)$ coming from the invariant $\overline{10}\cdot 126_1\ (126_1\overline{126}_1)$ contain a term linear in $\chi_{\bar{5}}$, and to this order it is the only such term in the potential! Hence it would have been unnatural to require that 126_1 have a vev along the SU(5) singlet only. At the same time the term linear in χ_{45} coming from this invariant is of $O(v^3)$ and there are other terms of this order in the potential to cancel against. Only $\chi_{\bar{5}}$ has a linear term of $O(vV^2)$. The presence of $\langle\bar{5}\rangle$ in 126_1 is crucial since it prevents the model from predicting too low a t-quark mass.

5. The beta function

In general, the symmetry breaking in a grand unified model based upon a gauge group G will proceed in several steps before arriving at the low-energy model $G_n \times SU(3)_c$ where G_n is the electroweak gauge group and is at least as large as $SU(2)_L \times U(1)$. After the ith step there will be a remaining unbroken gauge group G which is a valid symmetry up to a scale $(2M)$, where M is the mass of a gauge vector boson which is in G_{i-1} and not in G_i (note that we are writing the sequence of symmetry breaking as

$$G \to G_1 \to G_2 \to \cdots \to G_i \to \cdots \to [G_{fl} \times SU(3)_c]) . \tag{5.1}$$

In the SU(5) model there is only one possible pattern of symmetry breaking compatible with low-energy phenomenology: $SU(5) \to SU(3)_c \times SU(2)_L \times U(1)$. However, as soon as one considers larger groups, the possible patterns of symmetry breaking compatible with the world as we know it become more numerous. Such models offer some hope of partially filling the "desert" region between 300 GeV and 10^{15} GeV which is present in the simplest grand unified models based on SU(5).

In a generic SO(10) model there are a number of possible symmetry breaking patterns as illustrated in fig. 1 [15, 16].

In this section, as an illustration of a temporarily free model, we discuss the running of the various Yang-Mills couplings for the symmetry-breaking scheme

$$SO(10) \underset{54_H}{\to} SU(4) \times SU(2)_L \times SU(2)_R \underset{126_1}{\to} SU(3) \times SU(2)_L \times U(1)_Y \tag{5.2}$$

of our model which is realized when $\langle 54_H \rangle > \langle 126_1 \rangle$. We compute the renormalization of the $m_{-1/3}/m_{-1}$ mass relations down from the unification scale. We consider the effects of this analysis on the value of $\sin^2\theta_w$ and on the lifetime of the proton. Furthermore, since this model is only temporarily free, we compute the position of the Landau "singularity".

We call the scale at which SO(10) breaks down to $SU(4) \times SU(2)_L \times SU(2)_R$, m_2; the scale at which $SU(4) \times SU(2)_L \times SU(2)_R$ breaks down to $SU(3) \times SU(2)_L \times U(1)_Y$ is called m_1; and the scale at which $SU(3) \times SU(2)_L \times U(1)_Y$ breaks to $SU(3) \times U(1)_{EM}$ is called m_0. m_0 is related to the mass of the W boson through $m_0 \simeq 2M_W$. m_1 and m_2 are similarly related to the masses of the vector bosons that become massive at those scales.

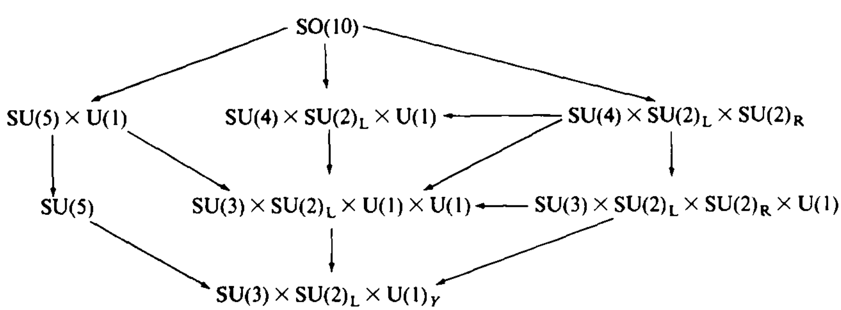

Fig. 1. Possible paths of symmetry breaking for the group SO(10). By suitable choice of Higgs representations any given sequence of symmetry breaking may occur that is consistent with the flow of the lines in this figure.

With forethought we choose to normalize the generators of SO(10) to 2 in the $\underset{\sim}{16}$ representation:

$$\mathrm{Tr}\left[T(\underset{\sim}{16})^2\right]=2. \tag{5.3}$$

This will give us the expression for the electric charge operator,

$$Q=T_{3\mathrm{L}}^{(10)}+\left(\tfrac{5}{3}\right)^{1/2}Y, \tag{5.4}$$

where $T_{3\mathrm{L}}^{(10)}$ is the diagonal $\mathrm{SU}(2)_{\mathrm{L}}$ generator [the superscript indicates that it is normalized as embedded in SO(10)] and Y is the $\mathrm{U}(1)_Y$ generator as embedded in SO(10), so as to conform with the traditional SU(5) expression. This latter point is assured if we normalize the generators of any SU(n) subgroup of SO(10) to $\frac{1}{2}$ in the $\underset{\sim}{n}$ representation,

$$\mathrm{Tr}\left[T^{(n)}(\underset{\sim}{n})^2\right]=\tfrac{1}{2}. \tag{5.5}$$

By tracking the three low-energy couplings up to the scale m_2 at which SO(10) first breaks, we will get three expressions that depend on the parameters m_1, m_2, and $\alpha_{10}(m_2)$ (α_{10} is the SO(10) gauge coupling squared divided by 4π). These can be determined by using as inputs the values of α_s, $\sin^2\theta_w$ and α_{EM}, all at m_0. The behavior of the couplings for a simple assumption about scalar thresholds is shown in fig. 2. We work with beta functions to lowest order in g and treat all mass thresholds in the theta function approximation.

We write g_{10} for the SO(10) gauge coupling; similarly we write g_4 for SU(4), $g_{2\mathrm{L}}$ for $\mathrm{SU}(2)_{\mathrm{L}}$, $g_{2\mathrm{R}}$ for $\mathrm{SU}(2)_{\mathrm{R}}$, g_Y for $\mathrm{U}(1)_Y$ and e for $\mathrm{U}(1)_{\mathrm{EM}}$. We use a similar notation for the quantities b that appear in the respective beta functions.

In general a Yang-Mills coupling g runs according to ($\mu<m$) [17]

$$g(\mu)^{-2}=g(m)^{-2}+2b\ln\frac{m}{\mu}, \tag{5.6}$$

assuming that there are no mass thresholds between μ and m and that the coupling remains perturbative in that region. The only difficulty in running, say, what starts as the SU(3) coupling, up to the unification scale is in determining the boundary conditions applicable as one goes from one region to an adjacent one. With the normalizations in eqs. (5.3) and (5.5) we have,

$$\begin{gathered}
g_3(m_1)=g_4(m_1), \qquad g_4(m_2)=g_{10}(m_2), \\
g_{2\mathrm{L}}(m_2)=g_{10}(m_2), \qquad g_{2\mathrm{R}}(m_2)=g_{10}(m_2), \\
g_Y(m_1)^{-2}=g_{\mathrm{R}}(m_1)^{-2}\sin^2\phi+g_4(m_1)^{-2}\cos^2\phi, \\
e(m_0)^{-2}=f^2\left[g_Y(m_0)^{-2}\sin^2\eta+g_{2\mathrm{L}}(m_0)^{-2}\cos^2\eta\right],
\end{gathered} \tag{5.7}$$

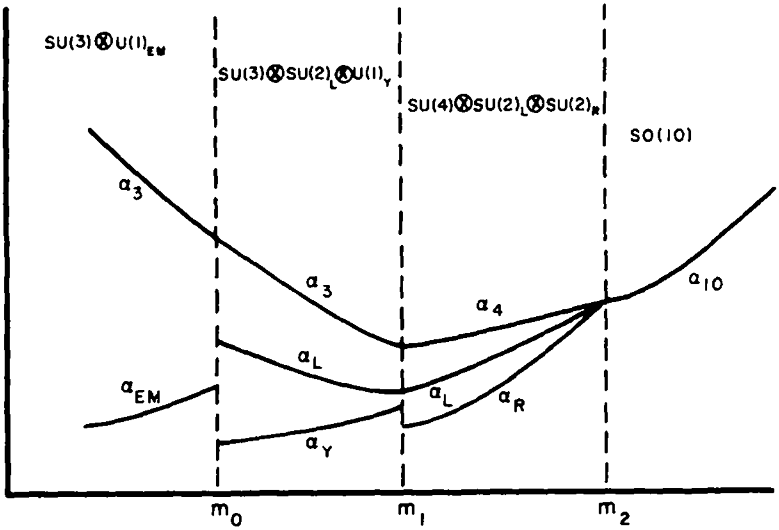

Fig. 2. Scaling behavior of the coupling constants.

and g_{2L} is continuous across the threshold at m_1. The qualitative behavior of these coupling constants for the breaking scheme (5.2) of our model is shown in fig. 2 $[\alpha_i \equiv g_i^2/4\pi]$.

In eq. (5.7), ϕ is a mixing angle which specifies which linear combination of the $SU(2)_R$ diagonal generator T_{3R} and the generator T_{15} in SU(4) becomes the $U(1)_Y$ generator. Similarly, η specifies which linear combination of T_{3L} and Y becomes Q and f is a constant which normalizes the electron charge to -1. The conditions that $Y=0$ for the SU(5) singlet fermion and that $Q=0$ for the $SU(2)_L$ doublet neutrino give

$$\sin\phi = \sqrt{\tfrac{3}{5}}\,, \qquad \cos\phi = \sqrt{\tfrac{2}{5}}\,,$$

$$\sin\eta = \sqrt{\tfrac{5}{8}}\,, \qquad \cos\eta = \sqrt{\tfrac{3}{8}}\,, \tag{5.8}$$

$$f = \sqrt{\tfrac{8}{5}}\,.$$

The expression for, say, g_Y^{-2} is obtained from the following argument (which can easily be generalized to cases more complicated than the linear combination of U(1)'s that we treat here) [18]. Say that we have vector fields A and B coupling respectively with generators X and Y and coupling constants g and g':

$$gAX + g'BY\,. \tag{5.9}$$

Then, if symmetry breaking leaves the combination $aX+bY$ unbroken (where

$a^2+b^2=1$), we have

$$gAX+g'BY=\tilde{g}C(aX+bY)+\hat{g}D(cX+dY), \tag{5.10}$$

where D is the vector which couples through the broken generator $cX+dY$, $\tilde{g}$ is the coupling constant for the vector field C and $\hat{g}$ is that for the vector D. This gives

$$gA=\tilde{g}aC+\hat{g}cD, \tag{5.11}$$

$$g'B=\tilde{g}bC+\hat{g}dD. \tag{5.12}$$

Then the orthonormality of A and B gives

$$g^2=\tilde{g}^2a^2+\hat{g}^2c^2, \qquad g'^2=\tilde{g}^2b^2+\hat{g}^2d^2, \qquad 0=\tilde{g}^2ab+\hat{g}^2cd, \tag{5.13}$$

from which follows

$$\tilde{g}^{-2}=a^2g^{-2}+b^2g'^{-2}. \tag{5.14}$$

It is from this that the last two boundary conditions in eq. (5.7) were obtained. [If the normalization conditions (5.3) or (5.5) are changed, the conditions in eq. (5.7) change accordingly. For example, if we choose $\mathrm{Tr}[T(16)^2]=\frac{1}{2}$ then we get $g_4(m_2)^2=\frac{1}{4}g_{10}(m_2)^2$.]

Using eqs. (5.6), (5.7) and the definition of the Weinberg angle,

$$W\equiv\frac{1}{\sin^2\theta_{\mathrm{w}}}=g_{2\mathrm{L}}(m_0)^2\left\{\frac{C^2}{g_{2\mathrm{L}}(m_0)^2}+\frac{1}{g_Y(m_0)^2}\right\}, \tag{5.15}$$

where $C^2=\frac{5}{3}$, we get the following three relations:

$$8\pi\{\tau+b_3y+(b_4-b_3)z\}=\alpha_3(m_0)^{-1}, \tag{5.16}$$

$$\begin{aligned}8\pi f^2\{\tau&+(b_{2\mathrm{L}}\cos^2\eta+b_Y\sin^2\eta)y\\&+[(b'_{2\mathrm{L}}-b_{2\mathrm{L}})\cos^2\eta+(b_{2\mathrm{R}}\sin^2\phi+b_4\cos^2\phi-b_Y)\sin^2\eta]z\}=\alpha_{\mathrm{EM}}(m_0)^{-1},\end{aligned} \tag{5.17}$$

$$\begin{aligned}(1+C^2-W)\tau&+[b_{2\mathrm{L}}(1-W)+C^2b_Y]y\\&+[(b'_{2\mathrm{L}}-b_{2\mathrm{L}})(1-W)+C^2(b_{2\mathrm{R}}\sin^2\phi+b_4\cos^2\phi-b_Y)]z=0,\end{aligned} \tag{5.18}$$

where

$$\tau\equiv[8\pi\alpha_{10}(m_2)]^{-1}, \qquad y\equiv\ln\frac{m_2}{m_0}, \qquad z\equiv\ln\frac{m_2}{m_1}. \tag{5.19}$$

b_{2L} is the $SU(2)_L$ value for b in (5.6) between m_0 and m_1 and b'_{2L} is the value between m_1 and m_2.

In calculating the various b's we use the expression obtained from the one-loop beta function [17]

$$b=\frac{-1}{16\pi^2}\frac{1}{r}\left\{\tfrac{11}{3}I_2(\underset{\sim}{\mathrm{Adj}})-\tfrac{2}{3}\sum_f I_2(\underset{\sim}{\mathrm{f}})-\tfrac{1}{6}\sum_s I_2(\underset{\sim}{\mathrm{s}})\right\}, \tag{5.20}$$

where r = rank of the group, and the second index of the representation $\underset{\sim}{r}$, $I_2(\underset{\sim}{r})$ is given by [19]

$$\frac{I_2(\underset{\sim}{r})}{r}=\mathrm{Tr}\left[T(\underset{\sim}{r})^2\right]. \tag{5.21}$$

The summations in eq. (5.20) are over all relevant left-handed fermion representations $\underset{\sim}{\mathrm{f}}$ and real scalar representations $\underset{\sim}{\mathrm{s}}$; complex scalar representations are counted twice since each complex scalar field contains two degrees of freedom; this comment also applies to doubled real or pseudoreal representations. $\underset{\sim}{\mathrm{Adj}}$ stands for the adjoint representation.

To treat the scalar and fermion thresholds it is convenient to consider the $SU(4)\times SU(2)_L\times SU(2)_R$ decomposition of the $\underset{\sim}{10}$, $\underset{\sim}{16}$, $\underset{\sim}{54}$ and $\underset{\sim}{126}$:

$$\begin{aligned}
\underset{\sim}{10}&=(\underset{\sim}{6},\underset{\sim}{1},\underset{\sim}{1})+(\underset{\sim}{1},\underset{\sim}{2},\underset{\sim}{2}),\\
\underset{\sim}{16}&=(\underset{\sim}{4},\underset{\sim}{2},\underset{\sim}{1})+(\underset{\sim}{\bar{4}},\underset{\sim}{1},\underset{\sim}{2}),\\
\underset{\sim}{54}&=(\underset{\sim}{1},\underset{\sim}{1},\underset{\sim}{1})+(\underset{\sim}{20},\underset{\sim}{1},\underset{\sim}{1})+(\underset{\sim}{6},\underset{\sim}{2},\underset{\sim}{2})+(\underset{\sim}{1},\underset{\sim}{3},\underset{\sim}{3}),\\
\underset{\sim}{126}&=(\underset{\sim}{6},\underset{\sim}{1},\underset{\sim}{1})+(\underset{\sim}{10},\underset{\sim}{3},\underset{\sim}{1})+(\underset{\sim}{\overline{10}},\underset{\sim}{1},\underset{\sim}{3})+(\underset{\sim}{15},\underset{\sim}{2},\underset{\sim}{2}).
\end{aligned} \tag{5.22}$$

In order to decide at which of the scales m_0, m_1 or m_2 a given Higgs obtains a mass we adopt the following ansatz: a scalar multiplet gets the largest possible mass consistent with its vacuum expectation value and the symmetry present at that scale. Thus a multiplet which gets no vacuum expectation value [such as the $(\underset{\sim}{6},\underset{\sim}{1},\underset{\sim}{1})$ in the complex $\underset{\sim}{10}$] will have a mass of $O(m_2)$. Certainly other assignments may be made since the only necessary constraints are that the $SU(3)\times SU(2)_L\times U(1)_Y$ theory be consistent with phenomenology at accessible energies and that the proton not decay too fast. Our choice is one with very few low-mass scalars.

As mentioned above, in the complex $\underset{\sim}{10}$ of Higgs the $(\underset{\sim}{6},\underset{\sim}{1},\underset{\sim}{1})$ has a mass $\sim m_2$; the $(\underset{\sim}{1},\underset{\sim}{2},\underset{\sim}{2})$ will have a mass $\sim m_0$. Since, by assumption, $\langle 54\rangle\sim m_2$, we put all of the masses of the (real) $\underset{\sim}{54}\sim m_2$. The representations 126_2 and 126_3 have vacuum expectation values $O(m_0)$ (in the 126_2 case it is along the SU(5) $\bar{5}$ direction and for the 126_3 it is along the 45 direction). In each case the vacuum expectation value is contained in the $(15, 2, 2)$. The SU(3) decomposition of the 15 of SU(4), $15 = 1^c + 3^c + \bar{3}^c + 8^c$, then shows that we will have a mass for the $(1^c, 2, 2)$ of $O(m_0)$ and masses for the remaining particles in $(15, 2, 2)$ of $O(m_1)$. The other particles in 126_2 and 126_3 may all be given masses of $O(m_2)$. For the 126_1 we have the addition of a vacuum expectation value along the SU(5) singlet direction $O(m_1)$. The SU(5) singlet is contained in the $(\overline{10}, 1, 3)$ which will therefore have a mass $\sim m_1$. Note that there is a baryon-number violating component in the $(\overline{10}, 1, 3)$. This has the possibility of putting a constraint on the scale m_1; this is discussed further below. These results are summarized in table 4.

With this information we can now compute b_Y, b_{2R}, b_{2L}, b_3 and b_4 (and b_{10}). With the normalizations (5.3) and (5.5) we get, for F families of fermions,

$$16\pi^2 b_Y = \tfrac{4}{3}F + \tfrac{4}{5}, \qquad 16\pi^2 b_{2L} = \tfrac{4}{3}F - \tfrac{20}{3}, \tag{5.23}$$

$$16\pi^2 b_{2R} = \tfrac{4}{3}F + \tfrac{44}{3}, \qquad 16\pi^2 b'_{2L} = \tfrac{4}{3}F + 8, \tag{5.24}$$

$$16\pi^2 b_3 = \tfrac{4}{3}F - 11, \qquad 16\pi^2 b_4 = \tfrac{4}{3}F + \tfrac{13}{3}, \tag{5.25}$$

$$16\pi^2 b_{10} = \tfrac{4}{3}F + 8. \tag{5.26}$$

For $F = 3$ these give

$$16\pi^2 b_Y = \tfrac{24}{5}, \qquad 16\pi^2 b_{2L} = -\tfrac{8}{3}, \qquad 16\pi^2 b_{2R} = \tfrac{56}{3}, \qquad 16\pi^2 b'_{2L} = 12, \tag{5.27}$$

$$16\pi^2 b_3 = -7, \qquad 16\pi^2 b_4 = \tfrac{25}{3}, \qquad 16\pi^2 b_{10} = 12. \tag{5.28}$$

TABLE 4
Scalars with masses less than or equal to the respective scales m_0, m_1 and m_2

Scale	Scalars	Multiplicity
m_0	$(1, 2, 2)$	2
	$(1^c, 2, 2)[\subset (15, 2, 2)]$	6
m_1	$(1, 2, 2)$	2
	$(\overline{10}, 1, 3)$	1
	$(15, 2, 2)$	6
m_2	10	2
	54	1
	126	3

The superscript c indicates that the SU(3) representation is being specified. In the remaining cases (except for m_2) the $SU(4) \times SU(2)_L \times SU(2)_R$ transformation properties are specified. For the case of m_2, the SO(10) representation is given.

Thus, in this scenario, the only couplings which diminish with increasing Q^2 are the SU(3) and the $SU(2)_L$ couplings in the region from m_0 to m_1.

The predictions for the ratio of the charged lepton mass to that of the charged $-\frac{1}{3}$quark mass are valid down to the scale m_1 where SU(4) is broken. To determine $m_{-1/3}/m_\ell$ at m_0 we must renormalize through [5,20,21]

$$\frac{m_{-1/3}(m_0)}{m_\ell(m_0)}=\frac{m_{-1/3}(m_1)}{m_\ell(m_1)}\left[\frac{\alpha_Y(m_0)}{\alpha_Y(m_1)}\right]^{3/4F}\left[\frac{\alpha_3(m_0)}{\alpha_3(m_1)}\right]^{12/(33-4F)}, \tag{5.29}$$

where F is the number of families. From eqs. (5.7) and (5.15) we find that

$$\alpha_Y(m_0)=f^2\left\{\cos^2\eta(W-C^2)^{-1}+\sin^2\eta\right\}\alpha_{\mathrm{EM}}(m_0). \tag{5.30}$$

If we write $m_{-1/3}(m_0)/m_\ell(m_0)=R(F)m_{-1/3}(m_1)/m_\ell(m_1)$, then, using eqs. (5.6), (5.8) and the two preceding equations, we find

$$R(F)=\left[1-8\pi b_3\alpha_3(m_0)(y-z)\right]^{12/(33-4F)}$$
$$\times\left[1-8\pi b_Y\alpha_{\mathrm{EM}}(m_0)\left\{\tfrac{3}{8}\left(W-\tfrac{5}{3}\right)^{-1}+\tfrac{5}{3}\right\}(y-z)\right]^{3/4F}. \tag{5.31}$$

Given $\alpha_{\mathrm{EM}}(m_0)$ and $\alpha_s(m_0)$, this expression depends on F through b_Y, b_3 and the exponents. Neither y nor z depend upon F. Aside from the explicit dependence upon W, R depends upon W implicitly through y and z. In this expression for R we are neglecting scalar thresholds.

To estimate the proton lifetime for vector induced decays in this scheme we use the result from BEGN [5] for the SU(5) model,

$$\Gamma(\mathrm{p}\to\text{lepton}+\text{any})=\frac{1}{\tau_{\mathrm{p}}}\simeq\frac{9\pi}{16}|\psi(0)|^2\frac{m_{\mathrm{q}}^2}{M_{\mathrm{X}}^4}(\alpha_{\mathrm{gum}})^{10/7}\left[\alpha_3(m_{\mathrm{proton}})\right]^{4/7}\left\{\tfrac{614}{9}\right\}, \tag{5.32}$$

for three families. For rough comparison we would wish to replace α_{gum} by $\alpha_{10}(m_2)$ and replace M_{X} by $\frac{1}{2}m_2$. We also note the existence of an additional weak doublet of baryon-number violating vector bosons in SO(10), thus providing a factor of 4 in the rate. Thus the ratio of the BEGN estimate to the present one is

$$\frac{1}{16}\left[\frac{\alpha_{\mathrm{gum}}}{\alpha_{10}(m_2)}\right]^{10/7}\left(\frac{m_2}{M_{\mathrm{X}}}\right)^4. \tag{5.33}$$

Using the SU(5) result $M_{\mathrm{X}}\simeq 4\times10^{14}$ GeV and $\alpha_{\mathrm{gum}}\simeq\frac{1}{40}$ the requirement that the proton not decay too fast implies that (5.33) be greater than one:

$$\frac{\tau_p(\mathrm{SO}(10))}{\tau_p(\mathrm{SU}(5))} \simeq \frac{(m_2)^4[\alpha_{10}(m_2)]^{-10/7}}{8\times 10^{61}} > 1\,. \tag{5.34}$$

In addition to requiring that this bound be satisfied, we have the additional constraint that the rate for scalar induced proton decay not be too large. We note that, since the mass of the relevant scalars is $O(m_1)$, this effect has the potential of being large. Two factors may act to reduce its magnitude. The first is that the scalars couple through a Yukawa coupling which is typically (m_N/m_1) times the gauge coupling. The second is that, since the 126_1 does not couple diagonally to the first family, any of its contributions to proton decay will be Cabibbo suppressed, occurring through the term $(A16_1\cdot 16_2)\cdot\overline{126}_1$ in (2.36). The Cabibbo suppression in the rate is then $\approx \sin^4\theta_c \simeq 0.002$. We may make some rough estimates of when we expect the scalar effects to become important in this model. For this we take the vector-induced proton decay rate to be roughly $\Gamma_v \sim \mathfrak{X} g^4 m_p^5/m_2^4$, where $\mathfrak{X}$ is a phase space factor typically $O(10^{-3}-10^{-4})$. In this language our scalar-induced proton decay rate is roughly $\Gamma_s \sim \mathfrak{X} g^4 (m_p^5/m_1^4)(m_N/m_1)^4\sin^4\theta_c$. Γ_s would begin to dominate Γ_v when $\Gamma_s/\Gamma_v \sim (m_N/m_1)^4(m_2/m_1)^4\sin^4\theta_c > 1$. If we use the light sector as a guide we might guess $m_N/m_1 \sim m_f/m_W \sim 10^{-2}-10^{-3}$, then $\Gamma_s/\Gamma_v > 1$ would correspond to $m_2/m_1 \gtrsim 10^3-10^4$. It is at this point that the scalar-induced process would begin to dominate the vector-induced one. To have the scalar-induced proton decay rate be less than the experimental limit we require $\Gamma_s > 10^{-30}$/year. With the above assumption for m_N/m_1 we get roughly $m_1 \gtrsim 10^{10}-10^{11}$ GeV.

It is unclear whether the appropriate parameter for a perturbation expansion is α or $\alpha\mathrm{Tr}(T^2)$ (where T is the generator in the representation relevant to a vertex in the diagram being considered) or any of a large number of other possibilities. Thus, to ask where, for a non-asymptotically free theory, the expansion parameter becomes non-perturbative is ambiguous. However, a measure of where perturbation theory fails that is independent of these questions is the position of the Landau "singularity." This is the value of m in eq. (5.6) for which $g(m)^{-2}$ vanishes. In the present model it is given by

$$\begin{aligned} m_L &= m_2 \exp[1/\{8\pi b_{10}\alpha_{10}(m_2)\}] \\ &= m_2\exp[\pi/\{6\alpha_{10}(m_2)\}]\,. \end{aligned} \tag{5.35}$$

The results of these calculations are given in figs. 2 through 5. (For these calculations, in computing m_W we have taken into account its variation with $\sin^2(\theta_w)$ given by

$$m_W^2 = \frac{\pi\alpha_{EM}}{\sqrt{2}\,G\sin^2\theta_w}\,. \tag{5.36}$$

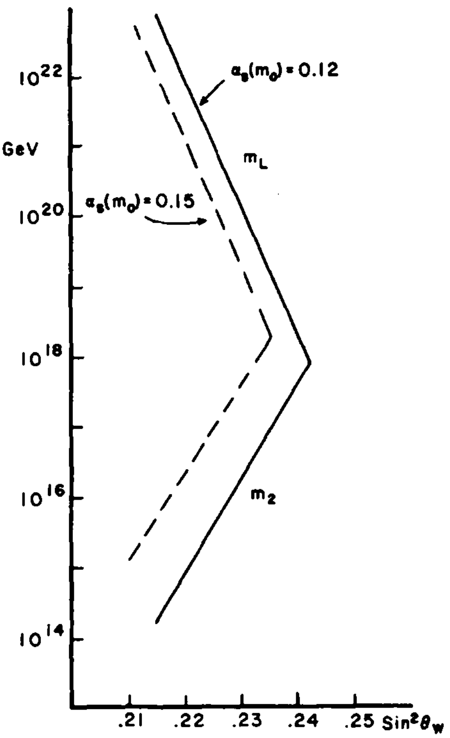

Fig. 3. m_2 and m_L as functions of $\sin^2\theta_w$.

This has a negligible effect on m_L, but does affect $\tau_p(SO(10))/\tau_p(SU(5))$ somewhat. In all calculations we have used $\alpha_{EM}(2m_W) \simeq \frac{1}{128}$. In fig. 3 we have plotted m_2 and m_L versus $\sin^2\theta_w$ for various values of the strong coupling constant $\alpha_s(=\alpha_3)$.

In fig. 4 we have plotted m_2 and m_1 versus $\sin^2\theta_w$. What should be noted from these graphs is that with our symmetry breaking scheme we have a lower bound for $\sin^2\theta_w$: the value for which $m_1 = m_2$. The inclusion of scalars in the calculation generally raises this value (in the absence of scalars the value is ~ 0.20 [3]). Independent of this is the fact that the value of $\sin^2\theta_w$ becomes larger as the difference between the scales (at which the symmetry breakings occur) grows.

In fig. 5 we show the values of $R(3)$ and $R(4)$ as a function of $\sin^2\theta_w$. These values are somewhat larger than those in the absence of scalars [3].

In fig. 6 we plotted the values of $\tau_p(SO(10))/\tau_p(SU(5))$ for vector-induced decays. It is clear that considering a model that is more complicated than the minimal SU(5) scheme can increase the proton's lifetime by many orders of magnitude. Since, for a reasonable assumption about the value of m_N/m_1, the scalar-induced proton decay rate becomes dominant for $m_2/m_1 \gtrsim 10^3$–10^4, the maximum value of $\tau_p(SO(10))/\tau_p(SU(5))$ occurs for $m_2/m_1 \sim 10^3$–10^4. Fig. 4 show that this generally occurs for $\sin^2\theta_w \sim 0.22$–0.23. Above this value $\tau_p(SO(10))/\tau_p(SU(5))$ decreases and

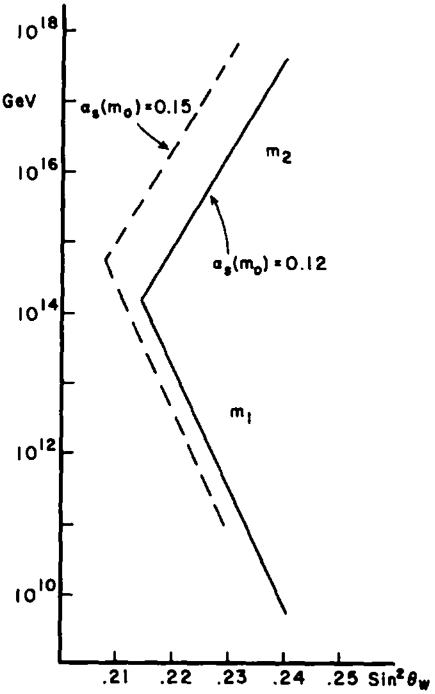

Fig. 4. m_1 and m_2 as functions of $\sin^2\theta_w$.

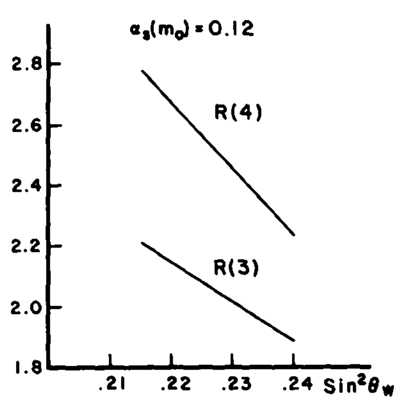

Fig. 5a. $R(3)$ and $R(4)$ as functions of $\sin^2\theta_w$ for $\alpha_s(m_0) = 0.12$.

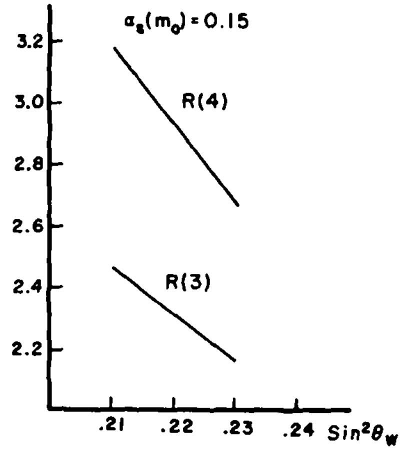

Fig. 5b. $R(3)$ and $R(4)$ as functions of $\sin^2\theta_w$ for $\alpha_s(m_0) = 0.15$.

would violate the experimental bound on the proton's decay rate if ever m_1 became less than $\sim 10^{10}$–10^{11} GeV. Note, however, that this value is generally the value near which $m_2 = m_L$ and thus, if we take the Landau "singularity" seriously, m_1 will be bounded below by roughly this value (this corresponds to the upper bound on $\sin^2\theta_w$ mentioned earlier). Thus it is possible that the Landau "singularity" could prevent the potential scalar induced proton decay from proceeding at too fast a rate. We mention, of course, that the relevant values of $\sin^2\theta_w$ are not the experimentally favored ones. It is worth noting in this context that the actual value of the proton

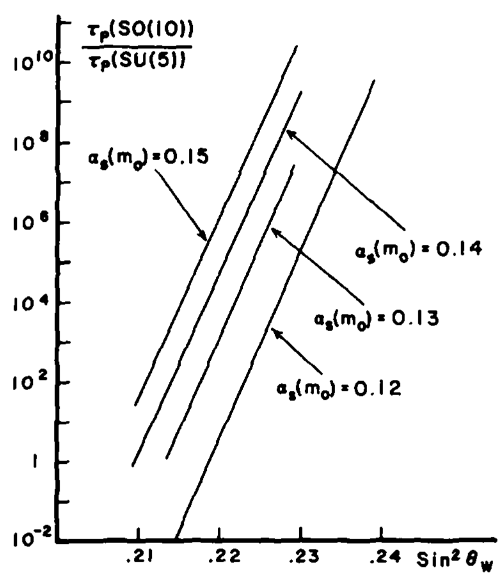

Fig. 6. $\tau_p(SO(10))/\tau_p(SU(5))$ as a function of $\sin^2\theta_w$.

lifetime is very sensitive to the values of the various input parameters. Thus, a calculation that uses only the lowest order beta function cannot be used to obtain precise results since higher order corrections can affect the results significantly. Furthermore, uncertainties in the location of scalar thresholds can also affect the results [22]*.

The lessons of this section show that one must be quite careful in making categorical statements about the specific numerical predictions of grand unified models**.

Appendix A

SO(10) SPINORS AND COUPLINGS

Explicit construction of the gauge and Yukawa couplings in SO(10) gauge models requires a study of how the various irreducible representations (irreps) are built up from products of the spinor irreps. Since there has been recent interest in the use of orthogonal groups for model building [5], we first present a general analysis for SO(N) [24] before specializing to SO(10).

A.1. SO(N) FOR N EVEN, $N=2n$

The spinor irreps of SO($2n$) are most easily studied by introduction of generalized Dirac γ matrices. We introduce $2n$ quantities Γ_i, $i=1,\ldots,2n$, that convert the basic quadratic form into the square of a linear form

$$X_1^2+X_2^2+\cdots+X_{2n}^2=(\Gamma_1X_1+\Gamma_2X_2+\cdots+\Gamma_{2n}X_{2n})^2. \tag{A.1}$$

This requires that

$$\{\Gamma_i,\Gamma_j\}=2\delta_{ij}. \tag{A.2}$$

(We indicate commutation by square brackets and anticommutation by wavy brackets.) For SO($2n$) the Γ_i may be represented by $2^n\times 2^n$ matrices.

If $R\in \mathrm{SO}(2n)$ then the $2^n\times 2^n$ matrices $S(R)$ such that

$$S(R)\Gamma_iS(R)^{-1}=R_{ij}\Gamma_j,\qquad i=1,\ldots,2n, \tag{A.3}$$

form a 2^n dimensional representation of SO($2n$) which we denote by Δ. The generators of this representation are given by

$$\Sigma_{ij}=\tfrac{1}{2}[\Gamma_i,\Gamma_j], \tag{A.4}$$

* This is independent of the possibility that an ansatz different from the one that we have chosen will affect the lowest order beta function significantly by having many more low-mass scalars.

** Calculations similar to those presented here have recently been given in ref. [23].

as can be seen by writing for an infinitesimal rotation $R_{ij}=\delta_{ij}+\omega_{ij}$, $\omega_{ij}=-\omega_{ji}$. We then have $S(R)\simeq 1+\frac{1}{2}i\omega_{ij}\Sigma_{ij}$, so that

$$S(R)\Gamma_i S(R)^{-1}\simeq\Gamma_i+i\omega_{kl}[\Sigma_{kl},\Gamma_i]=\Gamma_i+\omega_{ij}\Gamma_j=R_{ij}\Gamma_j, \tag{A.5}$$

where we have used (A.2) and (A.4). The generators in (A.4) satisfy the commutation relations

$$\left[\Sigma_{ij},\Sigma_{kl}\right]=i\left(\delta_{ik}\Sigma_{jl}+\delta_{jl}\Sigma_{ik}-\delta_{il}\Sigma_{jk}-\delta_{jk}\Sigma_{il}\right). \tag{A.6}$$

Since all the $S(R)$ commute with $\Gamma_{2n+1}=(-i)^n\Gamma_1\ldots\Gamma_{2n}$, Δ is reducible into two irreps, Δ_+ and Δ_-, of dimension 2^{n-1} formed by projection with $\frac{1}{2}(1\pm\Gamma_{2n+1})$. For n odd Δ_+ and Δ_- form complex irreps and are equivalent to the complex conjugate of each other. For n even Δ_+ and Δ_- form two inequivalent irreps with the complex conjugate of each being equivalent to itself. For n even, if we write $n=2m$, then for m even Δ_+ and Δ_- form real representations (also called orthogonal like representations) with

$$(\Delta_+\times\Delta_+)_S\supset 1,(\Delta_-\times\Delta_-)_S\supset 1, \tag{A.7}$$

where the subscript S indicates the symmetrized product while for m odd Δ_+ and Δ_- form pseudoreal representations (also called symplectic like representations) with

$$(\Delta_+\times\Delta_+)_A\supset 1,(\Delta_-\times\Delta_-)_A\supset 1, \tag{A.8}$$

where the subscript A indicates the antisymmetrized product.

The Γ matrices for SO($2n$) may be built up inductively from those of SO($2n-2$). A convenient basis in which Γ_{2n+1} is diagonal with $+1$ in the first n diagonal entries and -1 in the last n is given by

$$\begin{aligned}\Gamma_i&=\sigma_1\times\tilde{\Gamma}_i, \qquad i=1,2,\ldots,2n-2,\\ \Gamma_{2n-1}&=\sigma_1\times\tilde{\Gamma}_{2n-1},\\ \Gamma_{2n}&=\sigma_2\times 1,\\ \Gamma_{2n+1}&=(-i)^n\Gamma_1\cdots\Gamma_{2n}=\sigma_3\times 1,\end{aligned} \tag{A.9}$$

where $\tilde{\Gamma}_i$, $i=1,2,\ldots,2n-2$ are the Γ matrices for SO($2n-2$), $\tilde{\Gamma}_{2n-1}=(-i)^{n-1}\tilde{\Gamma}_1\ldots\tilde{\Gamma}_{2n-2}$ and σ_a, $a=1,2,3$, are the usual Pauli matrices. The induction starts at $n=1$ with $\Gamma_1=\sigma_1$, $\Gamma_2=\sigma_2$.

In the above representation of the Γ matrices the generators are given by

$$\Sigma_{ij}=\begin{pmatrix}\sigma_{ij} & 0\\ 0 & \tilde{\sigma}_{ij}\end{pmatrix}, \tag{A.10}$$

where σ_{ij} and $\tilde{\sigma}_{ij}$ are the $2^{n-1}\times 2^{n-1}$ matrix generators of Δ_+ and Δ_- respectively. In terms of $\tilde{\Sigma}_{rs}$ defined by

$$\tilde{\Sigma}_{rs}=\frac{1}{2i}\left[\tilde{\Gamma}_r,\tilde{\Gamma}_s\right], \qquad r,s=1,\ldots,2n-1,$$

we have

$$\sigma_{rs}=\tilde{\sigma}_{rs}=\tilde{\Sigma}_{rs}, \qquad r,s=1,\ldots,2n-1,$$

$$\sigma_{r2n}=-\tilde{\sigma}_{r2n}=\tilde{\Gamma}_r, \qquad r=1,\ldots,2n-1. \tag{A.11}$$

To construct the scalar, vector, 2nd rank tensor, etc. representations from the spinor representations, we consider a 2^n component covariant spinor ψ^a transforming under the representation Δ

$$\psi\to S\psi, \tag{A.12}$$

and a 2^n component contravariant spinor ϕ_a transforming under the contragredient representation $\hat{\Delta}$,

$$\phi\to(S^{-1})^{\mathrm{T}}\phi. \tag{A.13}$$

The representations $\hat{\Delta}$ and Δ are unitarily equivalent as is evidenced by the existence of a matrix C such that

$$CSC^{-1}=(S^{-1})^{\mathrm{T}}, \tag{A.14}$$

or using (A.3), $C\Gamma_i C^{-1}=\Gamma_i^{\mathrm{T}}$. In our representation of the Γ matrices $C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=C^{-1}$. The scalar is formed from ψ^a and ϕ_a as

$$\phi^{\mathrm{T}}\psi=\sum_{a=1}^{2n}\phi_a\psi^a, \tag{A.15}$$

while the vector is formed by

$$\phi^{\mathrm{T}}\Gamma_i\psi. \tag{A.16}$$

To see that this is indeed the vector note that under $\psi\to S\psi$, $\phi^{\mathrm{T}}\to\phi^{\mathrm{T}}S^{-1}$ we have

$$\phi^{\mathrm{T}}\Gamma_i\psi\to\phi^{\mathrm{T}}S^{-1}\Gamma_i S\psi=R_{ij}\phi^{\mathrm{T}}\Gamma_j\psi, \tag{A.17}$$

using eq. (A.3). The 2nd rank tensor is formed by

$$\phi^{\mathrm{T}}\Gamma_i\Gamma_j\psi, \qquad i\neq j, \tag{A.18}$$

etc.

Note that since $\psi\to S\psi$ we have

$$C\psi\to CS\psi=(S^{-1})^{\mathrm{T}}C\psi, \tag{A.19}$$

so that $C\psi$ transforms as a contravariant spinor and we may form the antisymmetric tensor representations as

$$\psi^{\mathrm{T}}C^{\mathrm{T}}\psi, \qquad \psi^{\mathrm{T}}C^{\mathrm{T}}\Gamma_i\psi, \qquad \psi^{\mathrm{T}}C^{\mathrm{T}}\Gamma_i\Gamma_j\psi, \qquad \text{etc.} \tag{A.20}$$

In general we have

$$\Delta\times\bar{\Delta}\sim T_0+T_1+\cdots+T_{2n}, \tag{A.21}$$

where T_i is the antisymmetric tensor representation of rank i. Any representation T_f gives rise to another representation T_{2n-f} through the SO($2n$) invariant relation

$$\alpha_{i_1\cdots i_{2n-f}}=\frac{1}{f!}\varepsilon_{i_1\cdots i_{2n-f}}{}^{j_1\cdots j_f}\alpha_{j_1\cdots j_f}, \tag{A.22}$$

where $\varepsilon_{i_1\cdots i_{2n}}$ is the totally antisymmetric symbol on $2n$ indices. We thus have $T_{2n-f}\sim T_f$ upon restriction to proper orthogonal transformations. For improper orthogonal transformations we have

$$T_{2n-f}\sim\sigma T_f, \qquad f=1,\ldots,n, \tag{A.23}$$

where $\sigma=+1$ for proper rotations and $\sigma=-1$ for improper rotations. The tensor representation T_n of dimension $\binom{2n}{n}$ splits up into two irreps T_n^+ and T_n^- of dimension $\frac{1}{2}\binom{2n}{n}$ according to the SO($2n$) invariant decomposition of a tensor of rank n, $\alpha=\alpha^++\alpha^-$, with

$$\alpha^{\pm}_{i_1\cdots i_n}=\tfrac{1}{2}\left(\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_nj_n}\pm\frac{1}{n!}\varepsilon_{i_1i_2\cdots i_nj_1j_2\cdots j_n}\right)\alpha_{j_1\cdots j_n}. \tag{A.24}$$

The representations T_n^+ and T_n^- are real for n even and the complex conjugates of each other for n odd. Upon splitting Δ into Δ_+ and Δ_- we have the following decomposition for the products appearing in $\Delta \times \hat{\Delta}$.

$$\left.\begin{aligned} \Delta_+ \times \Delta_+ &\sim T_0 + T_2 + \cdots \\ \Delta_- \times \Delta_- &\sim T_0 + T_2 + \cdots \\ \Delta_+ \times \Delta_- &\sim T_1 + T_3 + \cdots \end{aligned}\right\} (n \text{ even}); \tag{A.25}$$

$$\left.\begin{aligned} \Delta_+ \times \Delta_+ &\sim T_1 + T_3 + \cdots \\ \Delta_- \times \Delta_- &\sim T_1 + T_3 + \cdots \\ \Delta_- \times \Delta_+ &\sim T_0 + T_2 + \cdots \end{aligned}\right\} (n \text{ odd}). \tag{A.26}$$

A.2. SO(N) FOR N ODD, $N = 2n+1$

We take the $2n+1$ quantities Γ_i, $i = 1,\ldots,2n+1$, defined in (A.9) and define the representation $S(R)$ through the correspondence (A.3) as before but now with $i = 1,\ldots,2n+1$. This 2^n dimensional spinor representation Δ is irreducible and real (orthogonal like) for $n = 3,4,7,8,11,12,\ldots$ and pseudoreal (symplectic like) for $n = 5,6,9,10,13,14,\ldots$. The matrix C defined in (A.14) satisfies

$$\begin{aligned} C\Gamma_i C^{-1} &= \Gamma_i^{\mathrm{T}}, \qquad i = 1,\ldots,2n, \\ C\Gamma_{2n+1} C^{-1} &= (-1)^n \Gamma_{2n+1}^{\mathrm{T}}, \end{aligned} \tag{A.27}$$

and thus serves to relate covariant and contravariant spinors only for n even. For n odd the matrix $C\Gamma_{2n+1}$ must be used for this purpose. The various antisymmetric tensor representations are then constructed as before and we have

$$\Delta \times \Delta \sim T_0 + T_1 + \cdots + T_n. \tag{A.28}$$

A.3. SO(10)

In accordance with our previous discussion we take the ten 32×32 Γ matrices for SO(10) to be

$$\begin{aligned} \Gamma_1 &= \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_1, \\ \Gamma_2 &= \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_2, \\ \Gamma_3 &= \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_3, \\ \Gamma_4 &= \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_2 \times 1, \\ \Gamma_5 &= \sigma_1 \times \sigma_1 \times \sigma_1 \times \sigma_3 \times 1, \end{aligned}$$

$$\Gamma_6 = \sigma_1 \times \sigma_1 \times \sigma_2 \times 1 \times 1 ,$$

$$\Gamma_7 = \sigma_1 \times \sigma_1 \times \sigma_3 \times 1 \times 1 ,$$

$$\Gamma_8 = \sigma_1 \times \sigma_2 \times 1 \times 1 \times 1 ,$$

$$\Gamma_9 = \sigma_1 \times \sigma_3 \times 1 \times 1 \times 1 ,$$

$$\Gamma_{10} = \sigma_2 \times 1 \times 1 \times 1 \times 1 , \tag{A.29}$$

with $\Gamma_{11} = \sigma_3 \times 1 \times 1 \times 1 \times 1$. The generators for the 32-dimensional spinor representation are given by

$$\Sigma_{ab} = \frac{1}{2i}[\Gamma_a, \Gamma_b] = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix} . \tag{A.30}$$

Since SO(10) is rank 5, there are 5 diagonal commuting generators which form the Cartan subalgebra of SO(10). In our basis they are given by

$$\Sigma_{12} = 1 \times 1 \times 1 \times 1 \times \sigma_3 ,$$

$$\Sigma_{34} = 1 \times 1 \times 1 \times \sigma_3 \times \sigma_3 ,$$

$$\Sigma_{56} = 1 \times 1 \times \sigma_3 \times \sigma_3 \times 1 ,$$

$$\Sigma_{78} = 1 \times \sigma_3 \times \sigma_3 \times 1 \times 1 ,$$

$$\Sigma_{910} = \sigma_3 \times \sigma_3 \times 1 \times 1 \times 1 . \tag{A.31}$$

Upon projection with $\frac{1}{2}(1 \pm \Gamma_{11})$ the 32-dimensional spinor representation breaks up into two irreps, $\underset{\sim}{16}$ and $\overline{\underset{\sim}{16}}$, with $\overline{\underset{\sim}{16}}$ being equivalent to the complex conjugate of $\underset{\sim}{16}$. If we follow the conventional assignment of the left-handed fermion states to the $\underset{\sim}{16}$ then the $\overline{\underset{\sim}{16}}$ contains the right-handed *CP* conjugate states.

The product $\underset{\sim}{16} \times \underset{\sim}{16}$ then transforms as a Lorentz scalar and contains the SO(10) antisymmetric tensor representations of odd rank:

$$\underset{\sim}{16} \times \underset{\sim}{16} \sim T_1 + T_3 + T_5^+ = \underset{\sim}{10} + \underset{\sim}{120} + \underset{\sim}{126} ,$$

while the product $\overline{\underset{\sim}{16}} \times \underset{\sim}{16}$ transforms as a Lorentz vector and contains the antisymmetric tensor representations of even rank:

$$\underset{\sim}{16} \times \overline{\underset{\sim}{16}} \sim T_0 + T_2 + T_4 = \underset{\sim}{1} + \underset{\sim}{45} + \underset{\sim}{210} . \tag{A.32}$$

If ψ_L is a column vector containing the 16 left-handed two-component fermion fields, $\psi_L \sim \underset{\sim}{16}$, then the gauge covariant coupling is given by

$$\psi_L^+ \sigma^\mu D_\mu \psi_L, \tag{A.33}$$

with $\sigma^\mu = (1, \sigma^i)$ acting on Lorentz indices and $D_\mu = \partial_\mu + ig\frac{1}{2}\sigma_{ab}A_\mu^{ab}$, where $A_\mu^{ab} = -A_\mu^{ba}$, $a, b = 1, \ldots, 10$ are the 45 vector gauge fields.

The SO(6) ~ SU(4) subgroup of SO(10) is generated by σ_{mn}, $m, n = 1, \ldots, 6$. The SO(4) ~ $SU(2)_R \times SU(2)_L$ subgroup has the generators

$$T^i_{L(R)} = \tfrac{1}{8}\varepsilon^{ijk}\sigma^{6+j,6+k} \underset{(-)}{+} \tfrac{1}{4}\sigma^{6+i,10}, \qquad i, j, k = 1,2,3, \tag{A.34}$$

where with this normalization $T^3_{L(R)}$ is $\pm\frac{1}{2}$ when acting on a $SU(2)_{L(R)}$ doublet. The $SU(3)_c$ subgroup of SU(4) is generated by

$$\begin{array}{llll} \sigma_{12} + \sigma_{34}, & \sigma_{13} - \sigma_{24}, & \sigma_{14} + \sigma_{23}, & \sigma_{12} - \sigma_{26}, \\ \sigma_{16} + \sigma_{25}, & \sigma_{35} + \sigma_{46}, & \sigma_{36} - \sigma_{45}, & \sqrt{\tfrac{1}{3}}(\sigma_{12} - \sigma_{34} + 2\sigma_{56}), \end{array} \tag{A.35}$$

while the U(1) generator in SU(4) not in $SU(3)_c$ is given by

$$B - L = -\tfrac{1}{3}(\sigma_{12} - \sigma_{34} - \sigma_{56}). \tag{A.36}$$

The hypercharge, Y, is a linear combination of $B - L$ and T_3^R,

$$Y = -\tfrac{1}{2}(B - L) - T_3^R. \tag{A.37}$$

The electric charge is given as usual by $Q = T_3^L - Y$.

With these assignments and the representation of the Γ matrices given by (A.29), the $\underset{\sim}{16}$ of fermions is

$$\psi_L^T = (u_b, u_r, \nu, u_g, u_r^c, u_b^c, u_g^c, N^c, d_b, d_r, e, d_g, d_r^c, d_b^c, d_g^c, e^c)_L, \tag{A.38}$$

where b, r, and g are color labels and c denotes charge conjugation.

In order to write the Lorentz scalar, SO(10) invariant Yukawa couplings we first introduce a 32-component spinor $\Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$. If ϕ_a, $a = 1, \ldots, 10$ is a scalar transforming as $\underset{\sim}{10}$ under SO(10), then according to eq. (A.20) the Yukawa coupling of ϕ_a is given by

$$\Psi^T C^T \Gamma_a \sigma_2 \Psi \phi^a + \text{c.c.}, \tag{A.39}$$

where σ_2 is a Pauli matrix acting on Lorentz indices. The coupling to $\phi_{[a,b,c]} \sim \underset{\sim}{120}$ is given by

$$\Psi^{\mathrm{T}} C^{\mathrm{T}} \Gamma_a \Gamma_b \Gamma_c \sigma_2 \Psi \phi^{[a,b,c]} + \text{c.c.}, \tag{A.40}$$

while, following (21), the coupling of

$$\phi^{+}_{[abcde]} = \tfrac{1}{2}\left(\delta_{af}\delta_{bg}\delta_{ch}\delta_{di}\delta_{ej} + \frac{1}{5!}\varepsilon_{abcdefghij}\right)\phi_{[fghij]} \sim \overline{\underset{\sim}{126}}$$

is given by

$$\psi^{\mathrm{T}} C^{\mathrm{T}} \left(\tfrac{1}{2}(1-\Gamma_{11})\right)\Gamma_a\Gamma_b\Gamma_c\Gamma_d\Gamma_e\sigma_2\Psi\phi^{+}_{[abcde]} + \text{c.c.} \tag{A.41}$$

In the representation (A.29) we have

$$\Gamma_a = \begin{pmatrix} 0 & \gamma_a^{+} \\ \gamma_a & 0 \end{pmatrix}, \tag{A.42}$$

with γ_a a 16×16 matrix and γ_a^{+} the hermitian conjugate of γ_a. We can then write the couplings of ϕ_a, $\phi_{[abc]}$, and $\phi^{+}_{[abcde]}$ as

$$\begin{aligned} &\psi_{\mathrm{L}}^{\mathrm{T}}\gamma_a\sigma_2\psi_{\mathrm{L}}\phi_a, \\ &\psi_{\mathrm{L}}^{\mathrm{T}}\gamma_a\gamma_b^{+}\gamma_c\sigma_2\psi_{\mathrm{L}}\phi_{[abc]}, \\ &\psi_{\mathrm{L}}^{T}\gamma_a\gamma_b^{+}\gamma_c\gamma_d^{+}\gamma_e\sigma_2\psi_{\mathrm{L}}\phi^{+}_{[abcde]}. \end{aligned} \tag{A.43}$$

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