

HETEROTIC STRING THEORY* (II). The interacting heterotic string

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The theory of interacting heterotic strings is presented. Vertex operators are derived in both the bosonic and fermionic formulations of the theory and are shown to be consistent with gauge invariance, Lorentz invariance, and supersymmetry. Three- and four-point amplitudes for the scattering of massless string states are calculated and used to derive the low-energy field theory limit of the heterotic string. Divergences in string theories are discussed and it is shown that one-loop heterotic string amplitudes are finite and modular invariant only for gauge group $E_8 \times E_8$ or $\text{spin}(32)/Z_2$.

1. Introduction

In a previous paper [1] (hereafter referred to as (I)) we constructed a new theory of ten-dimensional superstrings as a chiral combination of the closed $D = 26$ bosonic and $D = 10$ fermionic strings. This heterotic string theory was shown to be $N = 1$ supersymmetric, Lorentz invariant and free of tachyons. Although containing only closed, orientable strings, it produces a gauge group G , which we argued could only be $E_8 \times E_8$ or $\text{spin}(32)/Z_2$. In one formulation of the theory, these gauge groups arise as a closed string generalization of the Kaluza-Klein mechanism [2]. The 16 gauge mesons associated with the Cartan subalgebra of G are related to the isometries of the sixteen-dimensional torus T^{16} (the maximal torus of G) on which the sixteen, left-moving, internal coordinates of the heterotic string lie. The additional 480 gauge mesons needed to complete the adjoint representation of G appear as massless solitons, which result when the closed string winds around T^{16} . We also presented an equivalent formulation of the theory, wherein the 16 internal bosonic coordinates were replaced by 32 fermionic coordinates with appropriate boundary conditions*.

We also constructed in (I) a manifestly reparametrization and Lorentz invariant action for the heterotic string. We showed that this action could be used to derive,

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* In both cases the mathematical object being represented is an affine Lie algebra. It is a matter of convenience which formalism one uses.

upon gauge fixing, the light-cone formulation of the theory which we used for quantization. All in all, a complete treatment of the free heterotic string was presented in (I).

In this paper we shall show that interactions can be introduced in the heterotic string theory without destroying any of its essential properties. This is not a trivial matter. It is certainly possible to construct consistent free string theories which are non-sensical at the interacting level. For example, one can easily construct bosonic string theories which are totally consistent in any dimension $d \leq 26$ by projecting the $d = 26$ theory into a d -dimensional subspace (of momentum space). However, this constraint is not geometrical and interactions will, at the one loop level, produce violations of Lorentz invariance and/or unitarity unless $d = 26$.

The basic idea behind the construction of the heterotic string was that the right-moving and the left-moving components of the string coordinates, bosonic or fermionic, are massless two-dimensional fields that do not mix as long as they propagate on orientable, closed two-dimensional world sheets. Thus we argued that the consistency of the heterotic string, which combines the right- and left-movers of two consistent string theories, is likely to be guaranteed by the consistency of its parents. In (I) this was demonstrated explicitly for the free heterotic string. This argument however should not be affected by string interactions since the split into right- and left-movers is geometrical and is not destroyed by interactions, which merely modify the topology of the two-dimensional manifolds on which the string coordinates propagate. In this paper we shall explicitly verify that, indeed, interactions do not affect the properties established for the free heterotic string—Lorentz invariance, supersymmetry, gauge invariance, absence of tachyons and positive metric Hilbert space.

In sect. 2 we show how interactions which describe the splitting and joining of the heterotic string are introduced, with emphasis on the geometrical nature of these interactions. Explicit expressions are given, in terms of operators acting on the Fock space states of the free string, that describe the amplitude for emission of massless particles from the string (the $N = 1$ supergravity multiplet and the $N = 1$ super Yang-Mills multiplet). These are the simplest vertex operators one can derive. They are sufficient for calculations of tree and one-loop amplitudes involving external massless particles as well as for the determination of the effective low-energy lagrangian describing these particles. We argue that with these vertices our interacting theory is Lorentz invariant, supersymmetric and $G (= E_8 \times E_8$ or spin $(32)/Z_2$) invariant.

Sect. 3 is devoted to a discussion of vertex operators in the fermionic formulation and an elucidation of the two-dimensional current algebra or affine Lie algebra on the string world sheet, which provides the gauge quantum numbers carried by the closed heterotic string.

Sect. 4 utilizes the vertex operators to establish the form of the effective field theory lagrangian that describes the low-energy (compared to the scale $\sqrt{T}$, T = string

tension) dynamics of the massless string excitations. This lagrangian is of great importance in attempts to discuss the phenomenology of the heterotic string. It can be used to check that proposed compactifications of ten-dimensional space are indeed solutions of the string equations of motion, and is an important tool in studying the theory at low energies. We also determine here the precise relation between the Yang-Mills coupling, Newton's constant and the string tension.

In sect. 5 we show how one can use the vertex operators to construct scattering amplitudes in the tree approximation. Here the full beauty of the heterotic string becomes apparent. As in all closed string theories there is a unique tree diagram, corresponding to a world sheet with the topology of a sphere, which describes all interactions and is completely symmetric (dual) in all the external particles. The expressions which exhibit this duality in the heterotic string, which unlike other closed string theories contains gauge bosons in addition to gravitons, are quite striking. We explicitly evaluate the tree amplitude for four external gauge bosons.

In sect. 6 we discuss loop amplitudes in general and present an explicit calculation of a simple one-loop diagram. Two issues are confronted in this section - that of the finiteness of the heterotic string and the possible existence of anomalies in global diffeomorphisms of the string world sheet [3,4]. There has been much confusion concerning the meaning of divergences that occur in some string theories. We therefore attempt to explain why string theories do not generate ultraviolet divergences, how the infinities that do occur (in open string theories) are related to the instability of the vacuum and why supersymmetric closed string theories (such as the heterotic string) are likely to be completely finite. The finiteness of the heterotic string is explicitly verified to one-loop order. We also address the question of the existence of global reparametrization anomalies that might occur at the one-loop level, and show that the requirement that these are absent for the heterotic string constrains the structure of the internal coordinates. In fact it is this requirement that finally restricts the gauge group to be of rank 16 and to possess a self-dual weight lattice, namely to be $\text{spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$.

We discuss the prospects for the heterotic string in sect. 7. The appendices contain an evaluation of some integrals needed in constructing tree amplitudes, and a discussion of the properties of the automorphic functions needed for heterotic loop amplitudes. Our notation will follow that of (I).

2. Heterotic string interactions

2.1. INTRODUCTION

It is relatively straightforward to incorporate interactions in a string theory. If one requires that the strings interact locally then there exist only a small number of possible interactions, which correspond to the splitting and joining of strings at points. Since the heterotic string is both closed and orientable it can have only one

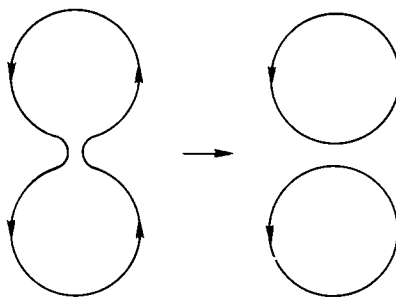


Fig. 1. Allowed interaction of closed orientable strings

type of interaction. This interaction (depicted in fig. 1) allows the closed string to split and form two closed strings whenever it self-intersects, or the inverse process. Type (I) (open, non-orientable) string theories, such as the $O(32)$ theory of Green and Schwarz, necessarily involve additional interactions. These correspond to open strings splitting (fig. 2a), or joining ends to form a closed string (fig. 2b), as well as self-interactions of open and closed non-orientable strings (figs. 2c and 2d). The absence of such processes for the heterotic string leads to a considerable simplification.

Heterotic string interactions are therefore totally determined by the amplitude for the string to split into two strings. This can be determined by geometrical considera-

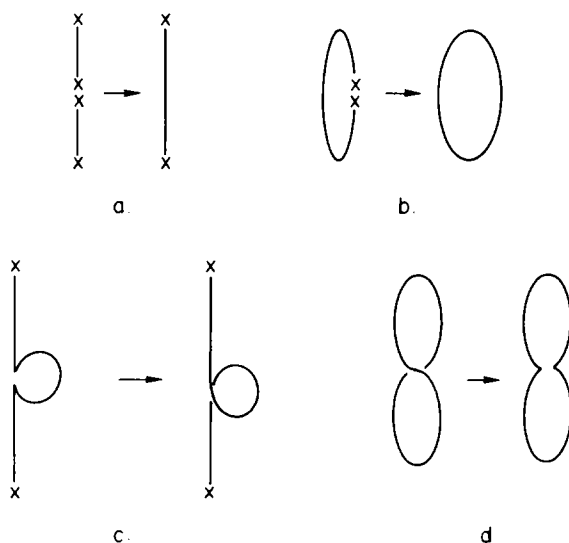


Fig. 2. Allowed interactions of open non-orientable strings.

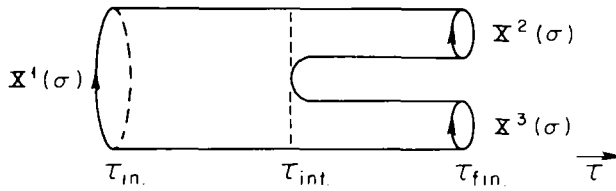


Fig. 3. World sheet representing the splitting of a closed string into two closed strings.

tions, most directly in the first quantized version of the theory. Here the dynamical variables are the spatial coordinates of the string, which vary as a function of time [5]. In the “light-cone gauge,” which we shall almost always employ, the string configuration is specified by $X^i(\sigma)$ ($i = 1 \dots 8$), and the light-cone coordinate $X^+ = \frac{1}{\sqrt{2}} (X^0 + X^9)$ is identified with the time parameter τ . In addition the superstring and the heterotic string contain internal degrees of freedom which, for the moment, we shall ignore. The propagating free string sweeps out a world sheet labelled by $X^i(\sigma, \tau)$, which, for a closed string, is a cylinder. The amplitude for a string to propagate from one configuration, $X_{in}^i(\sigma, \tau_i)$, at $\tau = \tau_i$ to another, $X_{fin}^i(\sigma, \tau_f)$, at $\tau = \tau_f$, is simply $\langle X_{fin}^i(\sigma) | e^{-iP(\tau_f - \tau_i)} | X_{in}^i(\sigma) \rangle$, where P is the generator of time (τ) translations. Alternatively this propagator can be written as a functional integral over string configurations $X^i(\sigma, \tau)$, which map out a cylinder between the initial and final string states, weighted by the classical action for $X^i(\sigma, \tau)$, which, in light-cone gauge, is that of $d - 2$ massless free fields.

From the functional integral point of view it is clear how to introduce interactions. In order to calculate the amplitude for a closed string to break up into two closed strings we should simply perform the functional integral over a surface with the topology of fig. 3 (the “pants diagram”) [6]. The “tree” approximation to the scattering of strings will be given by integrating over world sheets where the external string cylinders are attached to a sphere (fig. 4a); the N -loop amplitude will be obtained by attaching the cylinders to a sphere with N handles (fig. 4b). From the point of view of the first quantized theory the interaction of strings is introduced in a purely topological fashion, by enlarging the class of manifolds on which the two-dimensional massless fields, $X^i(\sigma, \tau)$, propagate. This is one of the reasons why string theories are bound to be renormalizable, since there is simply no way by which this interaction can be modified (more on this below).

Although in principle string scattering amplitudes could be calculated, to any desired order, by use of functional techniques, to date operator methods have been more widely used. Here one introduces an operator, acting on the direct product of single string Fock spaces, which yields the amplitude for the string to split. This can be deduced from the path integral representation, by considering, say the process described in fig. 3. Here the string propagates freely until the time τ_{int} , where the interaction takes place. The vertex operator is then given by the overlap integral

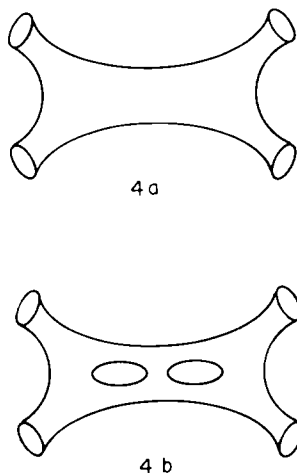


Fig. 4. World sheets for scattering of four strings at zero-loop (a) and two-loop (b) approximation.

between the initial and final states of the string at τ_{int} . In other words the matrix element of $V(\tau_{\text{int}})$, between states $\langle X^1(\sigma) |$ and $|X^2(\sigma)\rangle \times |X^3(\sigma)\rangle$ (where $|X^i(\sigma)\rangle$ are eigenstates of $X^i(\sigma)$ in the Fock space of the i th string), is given by an infinite product of delta functions that ensure that the curve $X^1(\sigma)$ coincides with the curve $X^2(\sigma) + X^3(\sigma)$.

The simplest of all vertices describes the emission of a zero-size pointlike string (fig. 5), from which the vertex for the emission of massless string states can be easily deduced. With this vertex one can derive the tree and one-loop amplitudes for the scattering of massless particles. These are sufficient, for example, to derive the leading low-energy field theory that describes the dynamics of the massless string modes. The full vertex is only required if one wishes to go beyond one-loop order. For a pointlike string the matrix element of the vertex [6] is simply proportional to

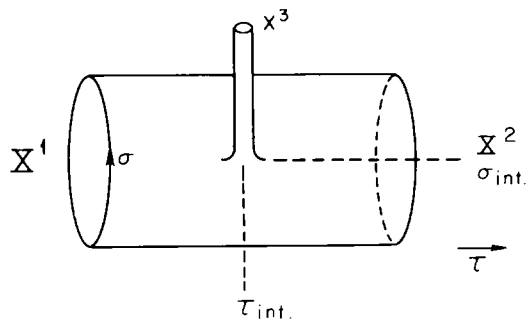


Fig. 5. Emission of a pointlike string.

$I_{12} \int_0^\pi d\sigma \delta[X^1(\tau_{\text{int}}, \sigma) - X^3]$, where I_{12} constrains $X^1(\tau_{\text{int}}, \sigma) = X^2(\tau_{\text{int}}, \sigma)$, and X^3 is the center of mass coordinate of string 3. Written as an operator in the Fock space of strings 1 and 2 (which we identify), I_{12} is simply the identity operator, and the vertex operator for the emission of a pointlike string with momentum k , at time τ , is proportional to

$$V(\tau, k, \lambda) = \int_0^\pi d\sigma W(\tau, \sigma, k, \lambda) e^{ik \cdot X(\tau, \sigma)}, \quad (2.1)$$

where W depends on the quantum numbers (internal and spacetime) of the emitted particle. The integral over σ reflects the fact that the particle can be emitted from any point σ on the string, $0 \leq \sigma \leq \pi$. The scattering amplitude can then be evaluated by sandwiching the S -operator

$$S = \int_{-\infty}^{\infty} d\tau_1 \dots d\tau_N T(V(\tau_1, k_1, \lambda_1) \dots V(\tau_N, k_N, \lambda_N)) \quad (2.2)$$

between appropriate asymptotic states (T denotes τ -ordering).

When one considers closed orientable strings the vertex operator V factorizes (inside the σ integral) into a product of V_R and V_L , where V_R (V_L) acts on right-moving (left-moving) modes of the string. This is a consequence of the fact that the string coordinates are massless free fields propagating on orientable surfaces, whose right- and left-movers do not interact with each other. This allows us to short cut the derivation of the vertices of the heterotic string. Aside from some group theoretical and numerical factors the vertices will be given as direct products of the right-moving superstring vertices and the left-moving bosonic string vertices, which have been derived previously. We shall in fact combine these to yield the massless particle emission vertices for the heterotic string. We see no obstacle to generalizing this construction to derive the general three string vertex, and thereby the second-quantized hamiltonian for the heterotic string.

2.2. REVIEW OF THE FREE HETEROTIC STRING

Let us first recall the kinematics of the free heterotic string [1]. The light-cone action is given by ($\alpha = \sigma, \tau$)

$$L = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left[\partial_\alpha X' \partial^\alpha X' + \partial_\alpha X' \partial^\alpha X'^I + i\bar{S}\gamma^I (\partial_\tau + \partial_\sigma) S \right], \quad (2.3)$$

where S is a ten-dimensional Majorana-Weyl spinor subject to the light-cone constraint $\gamma^+ S = 0$. The eight transverse coordinates $X^I(\tau - \sigma)$ ($I = 1 \dots 8$), together with S , comprise the right-moving physical degrees of freedom of the superstring [7]; whereas $X^I(\tau + \sigma)$ ($I = 1 \dots 8$) together with $X^I(\tau + \sigma)$ ($I = 1 \dots 16$) comprise the left-moving physical degrees of freedom of the bosonic string [5]. The coordinates X^I

lie on the maximal torus of $E_8 \times E_8$ or $\text{spin}(32)/Z_2$ and the X^I are coordinates on R^8 .

All these fields are periodic functions of $0 \leq \sigma \leq \pi$ and have the normal mode expansions (unless otherwise indicated we shall choose units in which $\alpha' = \frac{1}{2}$)

$$\begin{aligned} X'(\tau - \sigma) &= \frac{1}{2}x' + \frac{1}{2}p'(\tau - \sigma) + \frac{1}{2}i \sum_{n \neq 0} \frac{\alpha'_n}{n} e^{-2in(\tau - \sigma)}, \\ X'(\tau + \sigma) &= \frac{1}{2}x' + \frac{1}{2}p'(\tau + \sigma) + \frac{1}{2}i \sum_{n \neq 0} \frac{\tilde{\alpha}'_n}{n} e^{-2in(\tau + \sigma)}, \\ X^I(\tau + \sigma) &= x^I + P^I(\tau + \sigma) + \frac{1}{2}i \sum_{n \neq 0} \frac{\tilde{\alpha}^I_n}{n} e^{-2in(\tau + \sigma)}, \\ S^a(\tau - \sigma) &= \sum_n S^a_n e^{-2in(\tau - \sigma)}, \end{aligned} \quad (2.4)$$

with commutation relations

$$\begin{aligned} [x', p'] &= i\delta'^I, & [\alpha'_n, \alpha'_m] &= [\tilde{\alpha}'_n, \tilde{\alpha}'_m] = n\delta_{n+m,0}\delta'^{IJ}, \\ [\alpha'_n, \tilde{\alpha}'_m] &= 0, & \{S^a_n, \tilde{S}^b_m\} &= \frac{1}{2}(\gamma^{\cdot}(1 + \gamma_{11}))^{ab}\delta_{n+m,0}, \\ [\tilde{\alpha}^I_n, \tilde{\alpha}^J_m] &= n\delta_{n+m,0}\delta^{IJ}, & [x^I, P^J] &= \frac{1}{2}i\delta^{IJ}. \end{aligned} \quad (2.5)$$

The mass operator is

$$\frac{1}{4}(\text{mass})^2 = N + (\tilde{N} - 1) + \frac{1}{2} \sum_{I=1}^{16} (P^I)^2, \quad (2.6)$$

where

$$\begin{aligned} N &= \sum_{n=1}^{\infty} \left(\alpha'^I_n \alpha'_n + \frac{1}{2} n \tilde{S}^{\cdot}_n \gamma^{\cdot} S_n \right), \\ \tilde{N} &= \sum_{n=1}^{\infty} \left(\tilde{\alpha}^I_n \tilde{\alpha}'_n + \tilde{\alpha}^I_n \tilde{\alpha}^I_n \right) \end{aligned} \quad (2.7)$$

are the normal ordered number operators for the right- and left-moving modes, respectively.

The physical states are further constrained to satisfy $U(\Delta)|\text{phys}\rangle = |\text{phys}\rangle$, where $U(\Delta)$ is the operator which shifts σ by an amount Δ :

$$U(\Delta) = \exp 2i\Delta \left[N - \tilde{N} + 1 - \frac{1}{2} \sum_I (P^I)^2 \right]. \quad (2.8)$$

Finally we recall that the massless states of the heterotic string comprise a $D = 10$ supergravity multiplet consisting of $|i\rangle_R \times \tilde{\alpha}^I_{-1}|0\rangle_L$ and $|a\rangle_R \times \tilde{\alpha}^I_{-1}|0\rangle_L$, and a $D = 10$ super Yang-Mills multiplet consisting of the “neutral” bosons $|i\rangle_R \times \tilde{\alpha}^I_{-1}|0\rangle_L$ and fermions $|a\rangle_R \times \tilde{\alpha}^I_{-1}|0\rangle_L$ and the “charged” bosons $|i\rangle_R \times |P^I, (P^I)^2 = 2\rangle_L$ and fermions $|a\rangle_R \times |P^I, (P^I)^2 = 2\rangle_L$. Here $|i\rangle$ and $|a\rangle$ are the ground state of the right-handed sector of the superstring, and form a representation of the Clifford algebra of the fermionic zero mode operator S_0^a

$$\{S_0^a, S_0^b\} = [\gamma^{+1/2}(1 + \gamma_{11})]^{ab},$$

$$|a\rangle_R = \frac{1}{8}i(\gamma_i S_0)^a |i\rangle_R, \quad |i\rangle_R = \frac{1}{8}i(\bar{S}_0 \gamma_i)^a |a\rangle_R. \quad (2.9)$$

2.3 VERTICES

As we have discussed, the vertices for the heterotic string are given by products of right-moving superstring vertices [8] and left-moving bosonic string vertices [9]. Here we take these known components and form the heterotic vertices for the emission of the massless supergravity multiplet and the massless super Yang-Mills multiplet.

An on shell, light-cone gauge, supergravity multiplet is described by the polarization tensors $\rho^{\mu\nu}(k)$ for the bosons, and spinors $U^{\mu a}(k)$ for the fermions. Here

$$k^2 = 0, \quad \rho^{\mu\mu} = \rho^{\mu\nu} = k_\mu \rho^{\mu\nu} = k_\nu \rho^{\mu\nu} = 0,$$

$$U^{\mu a} = k U^\mu = k_\mu U^{\mu a} = \gamma^\mu U^\mu = \frac{1}{2}(1 - \gamma_{11})U^\mu = 0. \quad (2.10)$$

Thus $\rho^{\mu\nu}$ describes the graviton (traceless, symmetric $\rho^{\mu\nu}$), the antisymmetric tensor (antisymmetric $\rho^{\mu\nu}$) and the dilaton (the trace of $\rho^{\mu\nu}$); $U^{\mu a}$ contains the gravitino and a massless spinor.

The vertices for emission of these particles are then given by

$$V_B^{\text{grav}}(k) = \frac{4g}{\pi} \rho_{\mu\nu}(k) \int_0^\pi d\sigma B^\mu \tilde{P}^\nu e^{ik_\mu X^\mu(\tau, \sigma)},$$

$$V_F^{\text{grav}}(k) = \frac{4g}{\pi} \int_0^\pi d\sigma F^a \tilde{P}^\nu U^{\mu a}(k) e^{ik_\mu X^\mu(\tau, \sigma)}, \quad (2.11)$$

as long as $k^+ = 0$. The vertex for $k^+ \neq 0$ can be computed by boosting the above formula; the result is, however, unwieldy. Fortunately it is sufficient to consider the above case unless one wishes to calculate amplitudes with more than nine external particles. (A similar problem arises if one wishes to explore for anomalies in hexagon-loop amplitudes.) In fact one often reduces further by analytically continuing the amplitude to complex k^+ ; whereupon one can also set $k^- = 0$. Another simplification occurs if we restrict the polarizations to be transverse. As long as we

are confident of Lorentz invariance (whose validity we shall establish below) the results can then be assembled into unique covariant expressions. For all of the calculations we shall do in this paper it will be sufficient to consider transverse k^μ and polarizations, whereby we only need the transverse ($i = 1 \dots 8$) components of B^μ , $\tilde{P}^\mu$, etc. Thus

$$\begin{aligned} B^I &= P^I + \frac{1}{2} k^J R^{IJ}, \\ P^I &= \frac{dX^I}{d(\tau - \sigma)} = \frac{1}{2} p^I + \sum_{n \neq 0} \alpha_n^I e^{-2in(\tau - \sigma)}, \\ R^{IJ} &= \frac{1}{8} \bar{S} \gamma^{IJ} S, \\ \bar{F}^a &= \frac{1}{2} i (p^+)^{-1/2} [\bar{S} \gamma \cdot P - \frac{1}{6} : R^{IJ} k^J \bar{S} \gamma^I :]^a \end{aligned} \quad (2.12)$$

comes from the right-moving gravitational vertex of the superstring, and

$$\tilde{P}^I = \frac{1}{2} p^I + \sum_{n \neq 0} \tilde{\alpha}_n^I e^{-2in(\tau + \sigma)} \quad (2.13)$$

comes from the left-moving vector vertex of the bosonic string. [We use the notation $\gamma^{i_1 i_2 \dots i_n}$ for the totally antisymmetrized product of $\gamma^{i_1} \dots \gamma^{i_n}$.]

The vertices for the emission of massless gauge bosons are also easily constructed. Here we must distinguish the “neutral” gauge bosons, corresponding to the states $|i\rangle_R \times \tilde{\alpha}_{-1}^I |0\rangle_L$, and the charged gauge bosons which correspond to $|i\rangle_R \times |K^I, (K^I)^2 = 2\rangle_L$. These fill out the adjoint representation of $G = E_8 \times E_8$ or spin $(32)/Z_2$. The sixteen bosons with $K^I = 0$ correspond to the Cartan subalgebra of G ; the remaining bosons are labeled by their weights (K^I), which are the root vectors of G .

The vertices for the emission of neutral gauge mesons are essentially those of the gravitational multiplet, eq. (2.11), with the index ν being replaced by the internal index I . Indeed these sixteen bosons can be regarded as Kaluza-Klein gauge bosons arising from the $U(1)^{16}$ isometry of the internal torus. We thus have (for $k^- = 0$)

$$\begin{aligned} V_B^I &= \frac{4g}{\pi} \rho_\mu^I(k) \int_0^\pi d\sigma B^\mu \psi^I e^{ik_\mu X^\mu(\tau, \sigma)}, \\ V_F^I &= \frac{4g}{\pi} \int_0^\pi d\sigma F^{\mu\nu} \bar{\psi}^I U^{\mu I}(k) e^{ik_\mu X^\mu(\tau, \sigma)}, \end{aligned} \quad (2.14)$$

where $\rho_\mu^I (U^{\mu I})$ is the polarization vector (spinor) in the light-cone gauge, $k^2 = 0$;

$$k^\mu \rho_\mu^I = \rho^{\cdot I} = 0;$$

$$U^I = \gamma^+ U^I = \frac{1}{2}(1 - \gamma_{11})U^I = 0, \quad I = 1 \dots 16,$$

$$\psi^I = \frac{dX^I}{d(\tau + \sigma)} = P^I + \sum_{n \neq 0} \tilde{\alpha}_n^I e^{-2in(\tau + \sigma)}. \quad (2.15)$$

The vertex for the emission of the “charged” gauge bosons (and their supersymmetric partners) is of different form. The left-moving state here corresponds to the *tachyon* ground state of the bosonic string with momentum (= winding number) K^I in the internal dimension. Therefore the vertex or emission of the vector supermultiplet labeled by K^I is:

$$V_B^{K^I}(k) = \frac{4g}{\pi} \rho_\mu(k) \int_0^\pi d\sigma B^\mu e^{ik_\mu X^\mu(\tau, \sigma)} : e^{2iK^I \cdot X^I(\tau, \sigma)} : C(K_I),$$

$$V_F^{K^I}(k) = \frac{4g}{\pi} \int_0^\pi d\sigma F^a U^a(k) e^{ik_\mu X^\mu(\tau, \sigma)} : e^{2iK^I \cdot X^I(\tau, \sigma)} : C(K_I). \quad (2.16)$$

The factor $: e^{2iK^I \cdot X^I(\tau, \sigma)} :$ arises from the left-moving part of the “tachyon” vertex of the bosonic string. The factor of two in the exponential is due to the fact that $2K^I$ is the generator of translations in the internal left-moving space (see eq. (2.5)). The term must be normal ordered (for clarity we will sometimes use hats to refer to operators in the string Fock space)

$$: e^{2iK^I \cdot \hat{X}^I(\tau, \sigma)} : = \exp \left[-K^I \sum_{n=-\infty}^1 \frac{\tilde{\alpha}_n^I}{n} e^{-2in(\tau + \sigma)} \right] e^{2iK^I \cdot (\hat{x}^I + \hat{p}^I(\tau + \sigma))}$$

$$\times \exp \left[-K^I \sum_{n=1}^{\infty} \frac{\tilde{\alpha}_n^I}{n} e^{-2in(\tau + \sigma)} \right], \quad (2.17)$$

since $(K^I)^2$ does not vanish. The factor $e^{2iK^I \cdot \hat{x}^I}$ in the vertex shifts the internal momentum of a given string state by an amount K^I . Finally we require the operator cocycle $\hat{C}(K^I)$, which acts on left-moving string states with internal momentum P^I as [2]

$$\hat{C}(K^I)|P^I\rangle_L = \epsilon(K^I, P^I)|P^I\rangle_L. \quad (2.18)$$

This operator forms a projective representation of the translation group on the internal space

$$\hat{C}(K)\hat{C}(L) = \epsilon(K, L)\hat{C}(K+L). \quad (2.19)$$

where the factors $\epsilon(K, L)$ satisfy the two-cocycle conditions

$$\epsilon(K, L)\epsilon(K+L, M) = \epsilon(L, M)\epsilon(K, L+M). \quad (2.20)$$

This two-cocycle was used previously in our construction of free heterotic string. There we showed that the operators P^I and

$$E(K^I) = \int \frac{dz}{2\pi iz} : e^{2iK^I \hat{X}^I(z)} : \hat{C}(K^I) \quad (2.21)$$

(with $(K^I)^2 = 2$, $z = e^{2i(\tau + \sigma)}$) satisfied the Lie algebra of G (in the Chevalley basis). $E(K^I)$ is simply the part of the gauge boson vertex, evaluated at $k=0$, which commutes with the left-moving number operator.

We recall that the two cocycle $\epsilon(K, L)$ can be chosen to be bilinear. $\epsilon(K, L+M) = \epsilon(K, L)\epsilon(K, M)$, satisfies

$$\begin{aligned} \epsilon(K, L)\epsilon(L, K) &= (-1)^{K \cdot L}, \\ \epsilon(K, 0) &= -\epsilon(K, -K) = 1 \end{aligned} \quad (2.22)$$

and for $K^2 = L^2 = (K+L)^2 = 2$ coincides with the structure constants of G , which, in this basis, are all ± 1 . These properties will be useful in establishing G invariance of the interacting theory and in performing explicit calculations.

2.4. SYMMETRIES

The main advantage of working in light-cone gauge is that unitarity is manifest and the rules for calculating interactions are relatively simple. The main disadvantage is that manifest Lorentz invariance is lost and one must explicitly demonstrate that on-shell amplitudes are covariant. For the string this in fact restricts the dimension of spacetime to be 26 or 10 (for the fermionic or bosonic theories). Light-cone perturbation theory for the interacting bosonic string has been shown explicitly to be Lorentz invariant [6], and strong arguments have been given for the Lorentz invariance and supersymmetry of the superstring [7, 8].

In (I) we argued that the Lorentz invariance and supersymmetry of the light-cone gauge heterotic string is a consequence of the invariances of its fermionic and bosonic components. This is because the generators of Lorentz and supersymmetry transformations act separately on each sector of the heterotic string. Thus it was immediately obvious that the free heterotic string, whose Hilbert space consists of a direct product of the Fock spaces of the right-moving fermionic and the left-moving bosonic string (subject to a Lorentz invariant and supersymmetric constraint), yielded a unitary representation of the $D=10$, $N=1$ super Poincaré group. The same argument can be applied to the interacting heterotic string, since the vertex

operators are direct products of the component vertex operators of the two sectors. Thus the demonstration that these have the desired commutators with the generators of Lorentz and supersymmetry transformations is identical to the demonstration given for the superstring and for the bosonic string.

For example, consider supersymmetry. The heterotic string contains one supersymmetry, generated by

$$Q^a = i\sqrt{p^{+-}} (\gamma \cdot S_0)^a + 2i \frac{1}{\sqrt{p^{+-}}} \sum_n (\gamma_i S_{-n})^a \alpha_n^i. \quad (2.23)$$

This acts only on the right-moving modes on the string, which are those of the fermionic superstring. One must then show that this generator transforms V_B into V_F , and vice versa. This is a consequence of [8]

$$\left[\bar{\epsilon} \cdot Q, \rho_\mu(k) B^\mu e^{ik_\nu \cdot X_R^*(\tau - \sigma)} \right] = F^a(k) \frac{1}{2} \gamma_{\mu\nu} k^\mu \rho^\nu(k) \epsilon^a e^{ik_\nu \cdot X_R^*(\tau - \sigma)} \\ + \tau \text{ derivative term}, \quad (2.24)$$

$$\left[\bar{\epsilon} \cdot Q, F^a(k) U^a(k) e^{ik_\nu \cdot X_R^*(\tau - \sigma)} \right] = B^\mu(k) \cdot 2\bar{\epsilon} \left(\gamma_\mu - \frac{k^\mu \gamma^-}{k} \right) U(k) e^{ik_\nu \cdot X_R^*(\tau - \sigma)} \\ + \tau \text{ derivative term}, \quad (2.25)$$

which are not difficult to verify (at least for $k^+ = 0$). This ensures that S -matrix amplitudes constructed using these vertices are supersymmetric, since the τ derivative terms do not contribute when evaluated between on-shell states.

Finally we demonstrate that the above interaction vertices are G invariant. We must show that the vertices for emission of gauge bosons V^I and $V^{K'}$ yield an adjoint representation of the gauge group G . The generators of this group have been constructed above. They are (in the Chevalley basis) P^I and $E(K')$. Therefore we must verify the following commutation relations

$$\begin{aligned} [P^I, V_{B,F}^J] &= 0, \\ [P^I, V_{B,F}^{K'}] &= K^I V_{B,F}^{K'}, \\ [E(L'), V_{B,F}^J] &= L^J V_{B,F}^{L'}, \\ [E(L'), V_{B,F}^{K'}] &= \begin{cases} \epsilon(L, K) V_{B,F}^{L'+K'}, & (L+K)^2 = 2 \\ L^I \cdot V_{B,F}^I, & L+K=0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.26)$$

Since P^I and $E(L^I)$ act only on the internal coordinates the relevant pieces of $V_{B,F}^I$ and $V_{B,F}^{K^I}$ are K^I and $:e^{2iK^I \cdot X^I(\tau+\sigma)}: C(K^I)$, respectively.

The first two commutators are trivial, since P^I commutes with ψ^I and $[P^I, :e^{2iK^I \cdot X^I(\tau+\sigma)}:] = K^I :e^{2iK^I \cdot X^I(\tau+\sigma)}:.$ The third commutator follows from the fact that

$$\left[:e^{2iL^I \cdot X^I(\tau+\sigma)}:, \psi^I(\tau+\sigma) \right] = \delta(\sigma - \sigma') L^I :e^{2iL^I \cdot X^I(\tau+\sigma)}: \quad (2.27)$$

Finally the last commutator can be evaluated using the methods described in (I), where we derived the integrated version of this commutator.

Since the on-shell heterotic string states form unitary representations of the algebra of G generated by P^I and $E(K)$, this guarantees that S -matrix elements for the emission of massless bosons will be G invariant.

3. Current algebra and the fermionic representation of gauge interactions

The Yang-Mills interactions of the heterotic string can be interpreted in terms of a two-dimensional chiral current algebra defined on the world-sheet of the string [10]. This way of viewing the interactions has certain advantages, as we will now discuss. Each representation of the internal gauge symmetry of the heterotic string is just a particular realization of the current algebra. There exists a set of currents $j^A(\tau)$, $\tau_+ = \tau + \sigma$, belonging to the adjoint representation of the gauge group, $SO(32)$ or $E_8 \times E_8$, and obeying equal- τ_+ commutation relations:

$$\begin{aligned} [T_1 \cdot j(\tau_1), T_2 \cdot j(\tau_2)] &= i\delta(\tau_1 - \tau_2) [T_1, T_2] \cdot j(\tau_1) \\ &+ \frac{1}{4\pi} \delta'(\tau_1 - \tau_2) \text{tr}(T_1 T_2). \end{aligned} \quad (3.1)$$

(Here and below, we assume matrices T are in the vector representation of $SO(32)$ or an $SO(16) \times SO(16)$ subgroup of $E_8 \times E_8$; for the adjoint representation, divide each trace by 30.) By combining this current with the superstring vertex for emitting a boson or fermion ground-state, we obtain the light-cone gauge vertices for emitting gauge bosons and gauginos.

It is easiest to motivate the current-algebra approach in the fermionic representation of the gauge interactions, which realizes most or all of the gauge symmetries linearly. The fermionic realization of the current algebra consists of 32 real, free, chiral fermionic coordinates living on the string. Just as in the Neveu-Schwarz-Ramond formulation of the superstring [11], it is necessary to consider a sum of two "sectors" in the Hilbert space, with periodic and antiperiodic σ -boundary conditions on the fermions, and to project onto the subspace of states with even two-dimensional "fermion number," $(-1)^F = 1$, in order to obtain a consistent interacting

theory. The spectrum of states obtained in this way is identical to that of the bosonic formulation.

In the $SO(32)$ model, the 32 coordinates transform as a vector of $SO(32)$: the states in the antiperiodic sector are tensors of $SO(32)$, including the massless adjoint, while those in the periodic sector are spinors of $SO(32)$, starting with the massive, stable spinor. All of the $SO(32)$ currents are realized as fermion bilinears:

$$j^{IJ}(\tau, \sigma) = \sum_{n=-\infty}^{\infty} Q_n^{IJ} e^{-2in\tau} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2} \sum_{m=-\infty}^{\infty} \psi_n^I \psi_m^J \right) e^{-2in\tau}, \quad (3.2)$$

where m is integer for periodic boundary conditions and half-integer for antiperiodic boundary conditions. This current then provides the $SO(32)$ part of the gauge-particle vertex, for any of the $SO(32)$ generators, and clearly it is consistent with the $(-1)^F = 1$ projection imposed on the string spectrum.

In order to obtain the gauge group $E_8 \times E_8$ the 32 fermionic coordinates are split into two sets of 16 each, which are treated independently in imposing boundary conditions and making the projection onto even fermion number states. We have then a set of $SO(16) \times SO(16)$ currents realized as fermion bilinears, $j^{IJ}(\tau, \sigma)$ and $\tilde{j}^{IJ}(\tau, \sigma)$, and corresponding charges Q_n^{IJ} and $\tilde{Q}_n^{IJ}$. These charges generate an $SO(16)$ subalgebra of the E_8 affine Lie algebra; the currents form the gauge-group part of the vertex for emitting massless states in the $SO(16)$ subgroup of the corresponding E_8 , and amplitudes computed using the $SO(16)$ subgroup can be extended to the full E_8 by group invariance.

The full set of E_8 currents has a natural decomposition into $SO(16)$ representations [12], since the adjoint of E_8 decomposes into the adjoint (120) and spinor (128) of $SO(16)$. We would then like to construct the part of the E_8 currents $S^a(\tau, \sigma)$ which transform as the 128 of $SO(16)$, in terms of the 16 fermionic coordinates; the corresponding vertex operator would describe the emission of one of the massless $SO(16)$ spinor states in the sector with periodic boundary conditions. The charges obtained as Fourier components of S^a must take us from the Neveu-Schwarz (antiperiodic) sector with tensor representations to the Ramond (periodic) sector with spinorial representations, and back again. These operators change the boundary conditions obeyed by ψ (or $\tilde{\psi}$), and therefore cannot be expressed in closed form in terms of ψ (or $\tilde{\psi}$)*. The difficulty of constructing S^a and $\tilde{S}^a$ makes this representation somewhat less completely descriptive than the bosonic realization, but the linear realization of most of the symmetries more than overcomes this in certain applications.

We can approach the problem of constructing the spinor currents $S^a(\tau, \sigma)$ and $\tilde{S}^a(\tau, \sigma)$, and their associated vertex operators in several different ways. One way is to

* Likewise, in the $SO(32)$ case there is a vertex for emitting a massive stable spinor of $SO(32)$, which is not conveniently constructed in the fermionic picture.

attempt a construction similar to that of the fermion emission vertex in the Neveu-Schwarz-Ramond model [13], which would yield a fairly intractable expression in terms of ψ'_{NS} and ψ'_{R} . In this approach the operator $S^a(\tau, \cdot)$ introduces a cut in the fermion ψ' , which thereby changes its boundary condition and mode expansion from NS to R or vice versa; the expression for $S^a(\tau, \cdot)$ in ref. [13] is simply the contour integral defining the continuation of ψ' around the cut. Similarly, one could generalize the treatment of Friedan, Martinec and Shenker [14] of the fermion emission vertex in the covariant treatment of the Neveu-Schwarz-Ramond superstring, this approach, based on conformal field theory, would treat the fermion coordinates ψ' and the spinor current S^a which changes the boundary conditions on a more equal footing; both are conformal fields whose properties are determined solely in terms of their group representation content.

Before proceeding with a general discussion of the usefulness of current algebra, we explain how the currents are constructed in bosonic form. From what we said earlier about the relationship between the gauge-particle vertex and the gauge current, we can identify the current in the bosonic formulation. In the bosonic formulation only the generators of the Cartan subalgebra are realized linearly in the two-dimensional sense, as translations on the underlying torus. Their current (up to normalization) is just the string momentum operator for the extra sixteen dimensions:

$$\mathfrak{P}^I(\tau, \cdot) = \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^I e^{2in\tau, \cdot}. \quad (3.3)$$

The remaining currents are given by exponentials of the extra 16 coordinates, and labelled by momenta K^I , $(K^I)^2 = 2$, which are the charges of the currents with respect to the U(1) generators (3.3)

$$E_{K^I}(\tau, \cdot) = :e^{2iK^I X^I(\tau, \cdot)}: C(K^I). \quad (3.4)$$

Since X^I is the line integral of P^I , this representation is nonlocal in terms of the currents; in particular, a Fourier decomposition of the currents as in eq. (3.2) is inconvenient. The operators Q_n^A can be constructed as contour integrals of the current (3.4) (just as $Q_0^A = E(K^I)$ was constructed in (2.17)), but they do not have a simple expression in terms of the Fourier components of X^I .

Since the algebra of currents is independent of its realization, the difficulty in realizing some currents is not a problem for many of the applications of current algebra. The current algebra generally displays a greater symmetry than its most convenient realizations, and the commutation relations alone are sufficient for many simple applications, as we show below and in sect. 5. Some calculations, such as the spectrum of states of the theory and the corresponding traces in string-loop amplitudes, are much more straightforward in a coordinate realization. Others are more transparent in the fermionic representation of the gauge interactions. Because

of its close relationship to the current algebra, it provides a picture of the gauge interaction more accessible to particle physicists than the more geometrical bosonic formulation. More importantly, the linearly-realized currents make it easier to contemplate compactifications with curved space-time and gauge field expectation values. For instance, in the compactification schemes considered by Candelas et al. [15], which set the gauge field connection equal to the spin connection, the resulting string action is just a supersymmetric σ -model in the fermionic representation, which can be thoroughly analyzed. In the equivalent bosonic formulation, the supersymmetry would not be linearly realized. The fermionic representation is also extremely useful in analyzing more general background fields with non-zero antisymmetric tensor and dilaton expectation values [16]. We want to stress, however, that all realizations of the algebra are equivalent, and one may choose whichever is most suited to the problem at hand.

One application of current algebra is the computation of the gauge interaction part of scattering amplitudes in the light-cone gauge formalism. These computations are straightforward because the vertices are the currents themselves, while the asymptotic states which form the two ends of a tree diagram in light-cone gauge are simply constructed from the currents. Thus, after going to a propagator representation of the amplitude, which removes the τ_+ dependence of $j^A(\tau_+)$, we obtain

$$j^A(0) = \sum_{n=-\infty}^{\infty} Q_n^A, \quad (3.5)$$

where the Q_n^A obey

$$[T_1 \cdot Q_m, T_2 \cdot Q_n] = [T_1, T_2] \cdot Q_{m+n} + m\delta_{m+n,0} \frac{1}{2} \text{tr}(T_1 T_2). \quad (3.6)$$

Q_m for $m \geq 0$ annihilates the Fock-space vacuum $|0\rangle$, and $Q^A|0\rangle$ yields the left-moving part of a bosonic or fermionic gauge particle state:

$$|T_1, B \text{ or } F\rangle = T_1 \cdot Q_{-1}|0\rangle_{\text{left}} \times |B \text{ or } F\rangle_{\text{right}}. \quad (3.7)$$

Using (3.6) one can then evaluate, for example, the three-boson vector amplitude

$$\begin{aligned} \langle T_1, B_1 | V(T_1, B_2) | T_3, B_3 \rangle &= A_{\text{superstring}} \langle T_1 | V(T_2) | T_3 \rangle_{\text{left}} \\ &= A_{\text{superstring}} \langle 0 | (T_1 \cdot Q_{-1}) \left(\sum_{n=-\infty}^{\infty} T_2 \cdot Q_n \right) (T_3 \cdot Q_{-1}) | 0 \rangle_{\text{left}} \\ &= A_{\text{superstring}} \frac{1}{2} \text{tr}(T_1 [T_2, T_3]). \end{aligned} \quad (3.8)$$

One can apply the same technique to general tree diagrams with the additional

relation

$$\bar{z}^{\tilde{N}_0} Q_m = Q_m \bar{z}^{\tilde{N}_0 - m} \quad (3.9)$$

as we shall see in sect. 5.

One can try to go even further, replacing the bosonic or fermionic coordinate with currents as basic variables rather than as composite operators. In addition to their natural role in the vertex operators, one can define the gauge contribution to the string hamiltonian in terms of currents in the Sugawara form [17]:

$$H_{\text{gauge}} = \text{const} \sum_a : j^a(\tau_1) j^a(\tau_1) :. \quad (3.10)$$

The equivalence with the hamiltonian in the other pictures can be established by using a suitable limiting procedure to define the product of two currents (see, for instance, Coleman, Gross and Jackiw [18]). As indicated before, this approach is less natural in the usual formulation of string theory; in the conformal field-theory approach, which relies less heavily on explicitly constructing the spectrum of the theory, this approach is quite useful. In particular, $SO(9,1)$ current algebra can be used in calculating amplitudes in the superstring half of the theory [14, 19]. Thus the structure of both the left- and right-moving halves of the heterotic string is closely tied to current algebra, and group invariance deeply underlies the amplitudes of our theory.

4. Low-energy limit

String theories were originally developed as a phenomenological model of hadronic interactions. One persistent difficulty was the presence of massless states. The suggestion that string theory should be used to describe fundamental interactions including gravity was prompted by the realization that, in the limit of infinite string tension (or zero Regge slope) the interactions of the massless vector particles and massless tensor particles were precisely those of Yang-Mills gauge fields and gravitons respectively [20]. Type (I) superstring theories are known to give rise in the zero-slope limit to $N = 1$ super Yang-Mills theory coupled to supergravity in ten spacetime dimensions while type (II) superstrings reduce to one of two $N = 2$, $D = 10$ supergravity theories [21]. In this section we will discuss the low-energy limit of the heterotic string.

The low-energy effective point field theories which arise from string theories have proved useful in many regards. The anomaly cancellation mechanism of Green and Schwarz [22], although discovered by performing a string calculation, can be understood in terms of counterterms which may be added to the low-energy field theory. The fact that this mechanism worked not only for the gauge group $SO(32)$ but also for $E_8 \times E_8$ was realized at the field theory level before being understood in string theory. This low-energy field theory has also played an important role in understanding the constraints on proposed compactifications of ten-dimensional spacetime, in checking that the equations of motion are satisfied in these compactifi-

cations, and in studying mechanisms for obtaining supersymmetry breaking and a realistic fermion mass spectrum. In spite of these successes the low-energy field theory is probably only a small part of the story. Things that appear complicated and mysterious at the field theory level often have a simple explanation in string theory. As examples we cite the understanding of the Green-Schwarz anomaly cancellation in terms of modular invariance of the heterotic string (sect. 6), the finiteness of radiative corrections, and the understanding of the restrictions on allowed magnetic monopole charges in terms of the electric charge of string soliton states [23]. A full understanding of compactification and symmetry breaking will undoubtedly involve “stringy” considerations.

There are several methods by which one can construct the interactions that appear in the low-energy field theory. The method which is closest to the usual effective lagrangian approach in point field theories starts with the second-quantized string field theory and systematically integrates out all the massive modes in order to obtain an effective field theory describing the interactions of the massless modes. A second approach starts with the first-quantized string moving in background fields corresponding to coherent superpositions of the massless string modes. The action for the string is then that of a generalized non-linear sigma model and one can derive equations of motion for the background fields by demanding conformal invariance of the resulting two-dimensional field theory. This approach has been pursued in [16]. In this section we will use the older and more pedestrian approach of simply matching amplitudes for S -matrix elements calculated in the first-quantized string theory with those calculated in the corresponding low-energy field theory [20, 21].

The low-energy field theory is of course not renormalizable and should be regarded as a cutoff theory with cutoff of order the string mass scale $1/\sqrt{\alpha'}$. Higher derivative terms will appear scaled by appropriate powers of $\sqrt{\alpha'}$. Thus the field theory is an expansion in powers of external momenta times $\sqrt{\alpha'}$. Green and Schwarz’s analysis of anomaly cancellation [22] showed that such higher derivative terms can precisely cancel the anomalies in gauge and Lorentz transformations arising from fermion loops in the low-energy field theory. In open string theories there is no clear distinction between tree and one-loop string amplitudes. However in the closed heterotic string there is an unambiguous loop expansion and terms in the low-energy effective lagrangian which arise from a definite order in the loop expansion should be gauge and Lorentz invariant. At tree level this just requires that the antisymmetric tensor field $B_{\mu\nu}$ have the transformation properties given in ref. [22] in order to cancel the non-invariant parts of the gauge and Lorentz Chern-Simons terms which appear at tree level in the heterotic string. We have not carried through the low-energy expansion of one-loop amplitudes but we presume that they are also gauge and Lorentz invariant since they are finite and modular invariant (see sect. 6).

We will show that the Chapline-Manton action [24] for ten-dimensional $N = 1$ supergravity coupled to $N = 1$ Yang-Mills with the antisymmetric tensor field strength $H_{\mu\nu\lambda}$ modified by Chern-Simons terms, may be derived from the three-point

bosonic amplitudes of the heterotic string. We shall also determine the relationship between the Yang-Mills coupling g_{10} and the gravitational coupling κ . In addition we will show that there is a higher-order derivative term coupling the dilation field to $R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$. The presence of this term and the Lorentz Chern-Simons term and their coefficients relative to the corresponding terms involving the gauge fields play an important role in compactifications of the heterotic string. We have therefore striven to be careful and explicit with regard to our conventions and various numerical factors.

We will only compare the three-point bosonic couplings of the heterotic string with those of the low-energy field theory. The fermionic terms and higher point bosonic couplings are uniquely determined by supersymmetry, gauge invariance, and general coordinate invariance. Some four-point couplings will be discussed briefly in sect. 5.

The bosonic terms in the Chapline-Manton action are

$$\mathcal{L}_{\text{bos}} = \int d^{10}x e \left\{ -\frac{1}{2\kappa^2} R - \frac{1}{4} \phi^{-3/4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4} \phi^{-3/2} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{9}{16} \frac{\partial_\mu \phi \partial^\mu \phi}{\phi^2} \right\}, \quad (4.1)$$

where $\kappa = \sqrt{8\pi G}$ is the ten-dimensional gravitational coupling and

$$H_{\mu\nu\rho} = \frac{1}{2\sqrt{2}} \left(\partial_\mu B_{\nu\rho} - \frac{\kappa}{2\sqrt{2}} \text{tr} (A_\mu F_{\nu\rho} - \frac{2}{3} g_{10} A_\mu A_\nu A_\rho) + \text{cyclic perm.} \right) \quad (4.2)$$

is the field strength for the antisymmetric tensor field generalized to include the gauge field Chern-Simons three-form; g_{10} is the ten-dimensional gauge coupling constant. The trace is to be taken in the vector (32) representation of $\text{SO}(32)$ normalized so that $\text{tr } T_a T_b = 2\delta_{ab}$. This normalization corresponds to taking the roots to have length two as in (I). Rescaling the roots corresponds to rescaling the structure constants and hence rescaling the gauge coupling constant. Note that most physics literature uses a normalization corresponding to roots of length one so that $\text{tr } T_a T_b = \frac{1}{2} \delta_{ab}$ in the $\underline{n}$ of $\text{SU}(n)$ and $\text{tr } T_a T_b = \delta_{ab}$ in the $\underline{n}$ of $\text{SO}(n)$. For $\text{SO}(32)$ or $\text{E}_8 \times \text{E}_8$ we can convert the formulae that follow into traces in the adjoint representation denoted by Tr by replacing $\text{tr } T_a T_b$ by $\frac{1}{30} \text{Tr } T_a T_b$ everywhere.

In order to compare amplitudes it is convenient to first rescale the fields so that they have canonical kinetic energy terms and create properly normalized single particle states. Define $\phi = \exp(\sqrt{8/9} D)$ and rescale $B_{\mu\nu} \rightarrow \sqrt{2} B_{\mu\nu}$; we then have

$$\mathcal{L}_{\text{bos}} = \int d^{10}x e \left\{ -\frac{1}{2\kappa^2} R - \frac{1}{4} e^{\kappa D/\sqrt{2}} F_{\mu\nu}^a F^{\mu\nu a} - \frac{3}{2} e^{\kappa\sqrt{2} D} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{2} \partial_\mu D \partial^\mu D \right\} \quad (4.3)$$

with

$$H_{\mu\nu\rho} = \frac{1}{3} \left(\partial_\mu B_{\nu\rho} - \frac{1}{4} \kappa \operatorname{tr} \left(A_\mu F_{\nu\rho} - \frac{2}{3} g_{10} A_\mu A_\nu A_\rho \right) + \text{cyclic perm} \right). \quad (4.4)$$

If we also expand the metric as $g_{\mu\nu} = 2\kappa h_{\mu\nu}$, then $h_{\mu\nu}$ also has a properly normalized kinetic energy term and we can read off the various three-point couplings. Given this form of the effective action, various terms are easily compared with amplitudes calculated in the heterotic string with Fock space states of unit norm.

The three-point gauge field coupling is conventional. It can be calculated in the heterotic string using the vertex operators (2.14) and (2.16) with the polarization tensor normalized to unit magnitude. The non-vanishing amplitudes are those involving one neutral state and 2 “charged” states with non-zero lattice momentum, or those involving three charged states. The first gives the amplitude

$$A(I, K_1, K_3; k_i) = 4gK_3^I \delta_{K_1, -K_3} \delta\left(\sum k_i\right) \rho_\mu^1 \rho_\nu^2 \rho_\lambda^3 t^{\mu\nu\lambda} (1/2k_i), \quad (4.5)$$

where K_1 and K_3 are the lattice momenta, k_i , $i = 1, 2, 3$ are the spacetime momenta, chosen to be incoming, the ρ^i are the polarization vectors, I labels the neutral states and

$$t^{\mu\nu\lambda} (1/2k_i) = \frac{1}{2} \left(k_2^\mu \delta^{\nu\lambda} + k_3^\nu \delta^{\mu\lambda} + k_1^\lambda \delta^{\mu\nu} \right). \quad (4.6)$$

The amplitude for three charged states with lattice momenta K_i , $i = 1, 2, 3$, is

$$A(K_i; k_i) = 4g\epsilon(K_2, K_3) \delta_{K_1, -K_2-K_3} \delta\left(\sum k_i\right) \rho_\mu^1 \rho_\nu^2 \rho_\lambda^3 t^{\mu\nu\lambda} (1/2k_i). \quad (4.7)$$

These can both be written in the form

$$A(a, b, c; k_i) = g \rho_\mu^1 \rho_\nu^2 \rho_\lambda^3 f^{abc} \left[(k_2 - k_3)^\mu \delta^{\nu\lambda} + (k_3 - k_1)^\nu \delta^{\mu\lambda} + (k_1 - k_2)^\lambda \delta^{\mu\nu} \right], \quad (4.8)$$

where the f^{abc} are the structure constants in this basis, i.e. $f^{IK_1K_3} = K_3^I \delta_{K_1, K_3}$, $f^{K_1K_2K_3} = \epsilon(K_2, K_3) \delta_{K_1, -K_2-K_3}$. The identities (2.18) can be used to check that the amplitudes are totally symmetric in this basis. Comparing the above result to the usual Yang-Mills result shows that $g = g_{10}$ where g_{10} is the conventional ten-dimensional Yang-Mills coupling. As has been emphasized in ref. [25] the string tension and g are not independent parameters but are related through the dilaton vacuum expectation value $\langle D \rangle$. The result we quote for relations between couplings clearly corresponds to $\langle D \rangle = 0$. If we reinstate the string tension α' then since $[g] = [\text{length}]^0$ and $[g_{10}] = [\text{length}]^3$ we have $g = g_{10} (2\alpha')^{-3/2}$.

The relation between g and κ can be easily extracted from the graviton coupling to the dilaton energy momentum tensor (c.f. eq. (4.3))

$$\kappa h_{\mu\nu} (\partial^\mu D \partial^\nu D - \eta^{\mu\nu} \partial_\alpha D \partial^\alpha D). \quad (4.9)$$

The string amplitude corresponding to this coupling is easily evaluated by considering the graviton vertex (2.11) between normalized dilaton states $1/\sqrt{8} |i\rangle_R \times \tilde{\alpha}'^{-1} |0\rangle_L$ to give

$$A(1, 2, 3) = -\frac{1}{2} g \rho_2^{\mu\nu} [k_1^\mu k_3^\nu + k_1^\nu k_3^\mu], \quad (4.10)$$

where $\rho_2^{\mu\nu}$ is the graviton polarization tensor and k_1, k_3 the dilaton momenta. We thus have

$$g = 2\kappa(2\alpha')^{-2} \quad (4.11)$$

since $[\kappa] = [\text{length}]^4$. Therefore in the heterotic string the relation between the ten-dimensional gauge coupling and gravitational coupling is

$$\sqrt{2\alpha'} g_{10} = 2\kappa \quad (4.12)$$

as compared to $g_{10}^2 \sim \kappa\alpha'$ in open string theories. This relation survives compactification, since both g_{10} and κ are related to their four-dimensional values by the same factor of $\sqrt{V}$, where V is the volume of the compactified six-dimensional manifold. Thus we can determine the value of α' in terms of Newton's constant and the Yang-Mills coupling, i.e. $\alpha' = 2\kappa^2/g^2$. Since g is of order 1, α' is necessarily of order the Planck length.

The dilaton field D has trilinear couplings to A_μ and $B_{\mu\nu}$ which according to (4.3) are given by

$$\frac{1}{2\sqrt{2}} \kappa D (\partial_\mu A_\nu^\alpha \partial^\mu A^{\alpha\nu} - \partial_\nu A_\mu^\alpha \partial^\mu A^{\alpha\nu}), \quad (4.13)$$

$$3\sqrt{\frac{1}{2}} \kappa D (\frac{1}{3} \partial_\mu B_{\nu\rho} \partial^\mu B^{\nu\rho} + \frac{2}{3} \partial_\mu B_{\nu\rho} \partial^\nu B^{\rho\mu}). \quad (4.14)$$

These correspond to the on-shell matrix elements

$$\sqrt{\frac{1}{2}} \kappa (k_1 \cdot \rho_3) (k_3 \cdot \rho_1), \quad (4.15)$$

$$2\sqrt{2} \kappa \rho_1^\lambda \rho_3^\lambda k_1^\mu k_3^\nu, \quad (4.16)$$

respectively where k_1, k_3 are the gauge field or antisymmetric tensor field momenta and we have chosen a gauge in which $k^\mu \rho_\mu = k^\mu \rho^{\mu\nu} = 0$. These match the corresponding matrix elements calculated in the heterotic string with the dilaton vertex operator (2.11) (with $\rho_{\mu\nu} \rightarrow 1/\sqrt{8} \eta_{\mu\nu}$) evaluated between the appropriate states and with $g = 2\kappa$.

The gauge field Chern-Simons form in the antisymmetric tensor field strength leads to a trilinear coupling of the form

$$\kappa A_\mu^\alpha \partial_\nu A_\rho^\alpha (\partial^\mu B^{\nu\rho} + \partial^\nu B^{\rho\mu} + \partial^\rho B^{\mu\nu}), \quad (4.17)$$

which corresponds to the on-shell matrix element

$$-2\kappa \rho_1^\mu \rho_3^\nu \rho_2^{\alpha\beta} (\kappa_{3\alpha} k_{2\mu} \delta_{\nu\beta} + k_{1\alpha} k_{2\nu} \delta_{\mu\beta}). \quad (4.18)$$

Again this matches the corresponding amplitude calculated in the heterotic string.

We will now show that the heterotic string gives rise to additional trilinear couplings corresponding to the presence of the Lorentz Chern-Simons term in $H_{\mu\nu\rho}$ and to a dilaton coupling to $R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$. These can both be extracted from the general three “graviton” coupling. Evaluating the vertex operator (2.11) with polarization tensor $\rho_2^{\mu\nu}$ and momentum k_2 between states 1 and 3 with polarization tensors $\rho_1^{\alpha\beta}$, $\rho_3^{\lambda\rho}$ and momentum k_1, k_3 yields

$$4g\rho_1^{\alpha\beta}\rho_2^{\mu\nu}\rho_3^{\lambda\rho}t_{\alpha\mu\lambda}\left(\frac{1}{2}k_1\right)\left(t_{\beta\nu\rho}\left(\frac{1}{2}k_1\right)+\frac{1}{8}k_{2\beta}k_{3\nu}k_{1\rho}\right), \quad (4.19)$$

where $t_{\mu\nu\lambda}(k_i)$ is defined in eq. (4.6). The two factors entering this amplitude correspond to the usual Yang-Mills factor $t_{\alpha\mu\lambda}$ coming from the right-handed superstring and the Yang-Mills result with an $O(\alpha')$ correction coming from the left-handed bosonic string [19]. The $O(\alpha')$ correction precisely corresponds to the new terms we are interested in.

To obtain the coupling of $B_{\mu\nu}$ to the Lorentz Chern-Simons term we take $\rho_2^{\mu\nu}$ to be antisymmetric in which case (4.19) reduces to

$$\frac{1}{4}g\rho_1^{\alpha\beta}\rho_2^{\mu\nu}\rho_3^{\lambda\rho}\left[k_{2\rho}k_{1\nu}k_{3\alpha}k_{3\beta}\delta_{\mu\lambda}+k_{2\beta}k_{3\nu}k_{1\lambda}k_{1\rho}\delta_{\alpha\mu}\right]. \quad (4.20)$$

This corresponds to a term in the effective lagrangian

$$-\frac{1}{8}g\left(\partial_\alpha h_{\mu\beta}\partial^\beta\partial_\nu h_\rho^\alpha\right)\left(\partial^\mu B^{\nu\rho}+\partial^\nu B^{\rho\mu}\right). \quad (4.21)$$

Since $k_i \cdot k_j = 0$ on-shell this is equivalent to

$$\frac{1}{8}g\left(\partial_\beta h_{\mu\alpha}\partial^\beta\partial_\nu h_\rho^\alpha-\partial_\alpha h_{\mu\beta}\partial^\beta\partial_\nu h_\rho^\alpha\right)\left(\partial^\mu B^{\nu\rho}+\partial^\nu B^{\rho\mu}+\partial^\rho B^{\mu\nu}\right). \quad (4.22)$$

In the linearized approximation we have

$$e_\mu^a=\delta_\mu^a+\kappa h_\mu^a, \quad \omega_\alpha^{ab}=\kappa\left(h_\alpha^{a,b}-h_\alpha^{b,a}\right), \quad (4.23)$$

so that with $R_{\mu\nu}^{ab}=\partial_\mu\omega_\nu^{ab}-\partial_\nu\omega_\mu^{ab}+O(\omega^2)$, we can rewrite (4.22) as

$$\frac{g}{32}\frac{1}{\kappa^2}\omega_\mu^{ab}R_{\nu\rho}^{ab}\left(\partial^\mu B^{\nu\rho}+\partial^\nu B^{\rho\mu}+\partial^\rho B^{\mu\nu}\right). \quad (4.24)$$

Reinstating dimensions via $\frac{1}{8}g=\kappa^3/g_{10}^2$ we see that this corresponds to generalizing $H_{\mu\nu\rho}$ to include gauge and Lorentz Chern-Simons terms:

$$H_{\mu\nu\rho}=\frac{1}{3}\left[\partial_\mu B_{\nu\rho}-\frac{1}{4}\kappa\text{tr}\left(A_\mu F_{\nu\rho}-\frac{2}{3}g_{10}A_\mu A_\nu A_\rho\right)+\frac{\kappa}{4g_{10}^2}\text{tr}\left(\omega_\mu R_{\nu\rho}-\frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right)+\text{cyclic perm.}\right]. \quad (4.25)$$

Since ω_μ^{ab} and $R_{\nu\rho}^{ab}$ are already matrices in the vector representation of $SO(9,1)$ we have $\text{tr } \omega_\mu R_{\nu\rho} = \omega_\mu^{ab} R_{\nu\rho}^{ab}$.

We may also obtain the dilaton coupling to $R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$ by taking $\rho_1^{\alpha\beta}$ and $\rho_3^{\lambda\rho}$ to be symmetric and setting $\rho_2^{\mu\nu} = (1/\sqrt{8})\eta^{\mu\nu}$ in eq. (4.19). This yields

$$-\frac{1}{4}\sqrt{\frac{1}{8}}g\rho_1^{\alpha\beta}k_{3\alpha}k_{3\beta}\rho_3^{\lambda\rho}k_{1\lambda}k_{1\rho}, \quad (4.26)$$

where we have used $k_{i\mu}\rho_i^{\mu\nu} = 0$ and $\sum k_i = 0$. This corresponds to a term in the action

$$-\frac{1}{8}\sqrt{\frac{1}{8}}gDh_{\alpha\beta,\lambda\rho}h^{\lambda\rho,\alpha\beta} \quad (4.27)$$

in the linearized approximation. The general coordinate invariant generalization of this coupling must involve the dilaton coupling to a linear combination of R^2 , $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$. With $h_{\mu\nu}$ traceless and $\partial_\mu h_{\mu\nu} = 0$ only the $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$ term contributes to the on-shell three-point coupling and its contribution reduces to $4\kappa^2 h_{\nu\lambda,\mu\rho}h^{\mu\rho,\nu\lambda}$. Using $\frac{1}{8}g = \kappa^3/g_{10}^2$ in (4.27) is equivalent at the linearized level to

$$\frac{e^{-\kappa D/\sqrt{2}}}{8g_{10}^2}R_{\alpha\beta\lambda\rho}R^{\alpha\beta\lambda\rho}. \quad (4.28)$$

Finally if we choose ρ_1 and ρ_2 to be antisymmetric and ρ_3 to be symmetric we determine the couplings of $B_{\mu\nu}$ to the gravitational field. In addition to the standard term arising from the kinetic energy of $B_{\mu\nu}$, $h_{\alpha\beta}H_{\mu\nu}^\alpha H^{\beta\mu\nu}$, there arises a new term with two more derivatives. This term, not present in the Chapline-Manton action, cannot be totally fixed at this stage in the construction of the effective action. It corresponds to a linear combination of $R_{\mu\nu\sigma\lambda}H^{\mu\nu\alpha}H^{\sigma\lambda}_\alpha$ and $\nabla_\mu H_{\nu\alpha\beta} \cdot \nabla^\mu H^{\alpha\beta}$. In order to determine the relative coefficients of these terms one would have to evaluate $\mathcal{L}$ to next order in the α' expansion. We shall therefore not include it below.

We find the leading bosonic terms in the low-energy effective lagrangian by combining the previous results. These terms are particularly simple if we scale out the coupling constants by taking $A_\mu \rightarrow A_\mu/g_{10}$ and $B_{\mu\nu} \rightarrow \kappa B_{\mu\nu}/g_{10}^2$. We then obtain

$$\mathcal{L}_{\text{bos}} = \int d^{10}x e \left\{ -\frac{1}{2\kappa^2}R - \frac{1}{8g_{10}^2}e^{-\kappa D/\sqrt{2}}(\text{tr } F_{\mu\nu}F^{\mu\nu} - \text{tr } R_{\mu\nu}R^{\mu\nu}) \right. \\ \left. - \frac{1}{2}\partial_\mu D\partial^\mu D - \frac{3\kappa^2}{2g_{10}^4}e^{-\kappa D/\sqrt{2}}H_{\mu\nu\rho}H^{\mu\nu\rho} \right\}, \quad (4.29)$$

with

$$H_{\mu\nu\rho} = \frac{1}{3} \left[\partial_\mu B_{\nu\rho} - \frac{1}{4}\text{tr}(A_\mu F_{\nu\rho} - \frac{2}{3}A_\mu A_\nu A_\rho) \right. \\ \left. + \frac{1}{4}\text{tr}(\omega_\mu R_{\nu\rho} - \frac{2}{3}\omega_\mu \omega_\nu \omega_\rho) + \text{cyclic perm.} \right]. \quad (4.30)$$

The symmetry between gauge field terms and gravitational terms in the low-energy field theory of (4.29) has dramatic consequences for compactifications of the heterotic string. Compactification of the heterotic string to four dimensions by taking six of the coordinates to parametrize a flat 6-dimensional torus is presumably consistent but phenomenologically uninteresting since it leads to $N = 4$ supersymmetry and left-right symmetric fermions. In order to obtain chiral fermions in four dimensions it appears necessary to have a non-zero gauge field strength in the six compact dimensions.

In general, coupling the heterotic string to background gauge or gravitational fields leads to inconsistencies at the string level. The coupling to background fields is easily described in the fermionic formulation of sect. 3. The left-moving fermions couple to the gauge field pulled back to the string world sheet, $A_\alpha = (\partial X'/\partial \xi^\alpha) A_i$, $\xi_\alpha = (\sigma, \tau)$, while the right-moving fermions couple to the pullback of the spin connection $\omega_\alpha = (\partial X'/\partial \xi^\alpha) \omega_i$. In order to describe consistent string propagation the resulting nonlinear sigma model must be conformally invariant. Imbedding the spin connection in the gauge group leads to a left-right symmetric treatment of the fermions coupled to non-zero background fields. This ensures the absence of potential anomalies [26] and leads to 1 + 1 dimensional $N = 2$ or $N = 1$ supersymmetry depending on whether the compact space is Kähler or not. The arguments of ref. [27] then imply that the resulting sigma model has vanishing β function and hence is conformally invariant.

The heterotic string can also cure the problem of obtaining a solution to the classical equations of motion if the spin connection on the compact manifold is embedded in the gauge group, due to the coupling of the dilaton to $\text{tr}(R_{\mu\nu}R^{\mu\nu})$ [42, 15]. In fact it is clear from eq. (4.29) that the dilaton tadpole will vanish if

$$\frac{1}{30} \text{Tr} F_{\mu\nu} F^{\mu\nu} = \text{tr} R_{\mu\nu} R^{\mu\nu}. \quad (4.31)$$

For gauge group $O(32)$ and compactification on a $2n$ -dimensional manifold with $O(2n)$ holonomy the spin connection can be viewed as an $O(2n)$ gauge field and the minimal imbedding of $O(2n)$ in $O(32)$ with $\underline{32} = \underline{2n} + \text{singlets}$ clearly satisfies $\frac{1}{30} \text{Tr} F^2 = \text{tr} F^2 = \text{tr} R^2$. If the manifold is Kähler the holonomy group is $SU(n)$ or $SU(n) \times U(1)$ and the imbedding $\underline{32} = \underline{n} + \bar{\underline{n}} + \text{singlets}$ again satisfies (4.31). For gauge group $E_8 \times E_8$ such an imbedding also satisfies (4.31). To see this imbed the spin connection in an $SO(8)$ subgroup of E_8 and use the decomposition $E_8 \supset SO(8) \times SO(8)$ with $\underline{248} = (28, 1) + (1, 28) + (8_v, 8_v) + (8_s, 8_s) + (8'_s, 8'_s)$ where 8_v , 8_s , and $8'_s$ refer to the vector, spinor, and spinor' representations of $SO(8)$ respectively. For $SO(8)$ the quadratic Casimir in the $\underline{28}$ is 6 times that for the 8_v , 8_s , or $8'_s$ so $\frac{1}{30} \text{Tr} F^2 = \text{tr} F^2$ where the second trace is in the vector representation of $SO(8)$ and again (4.31) is satisfied.

5. Tree amplitudes

In this section we shall show how to use the vertex operators constructed above to calculate the tree scattering amplitudes of the heterotic string. As an illustration we shall explicitly evaluate the four gauge-boson scattering amplitude. This example will exhibit the extraordinary duality properties of the heterotic string. Since there is a unique world sheet that describes a given N -point tree amplitude with the topology of a sphere with N cylinders attached, all tree processes are contained in the same diagram. Thus the four point amplitude is given by fig. 4a, and describes string states being exchanged in all (s , t , and u) channels. Moreover, since heterotic gauge bosons are also described by closed strings, the amplitude described by fig. 4a will include both gauge and gravitational exchanges. This is in contrast to open string theories, in which gauge boson exchange occurs in the tree approximation, whereas graviton (or closed string) exchange is dual to an open string-loop diagram (as illustrated in fig. 6).

In the light-cone gauge formalism the N -particle scattering amplitude is calculated by evaluating the matrix element between asymptotic string states of the τ -ordered product of vertices for string emission, for a given τ -ordering, and then summing over all τ -orderings [6, 28]. Thus corresponding to the process depicted in fig. 7, where strings 2, 3, ..., $N - 1$ are emitted at times $\tau_2, \tau_3, \dots, \tau_{N-1}$, we have the contribution

$$A(1, 2, \dots, N) \sim \int \prod_{\substack{i=2 \\ \tau_i < \tau_{i-1}}}^{N-1} d\tau_i \langle N | V_{N-1}(k_{N-1}, \tau_{N-1}) \dots V_2(k_2, \tau_2) | 1 \rangle, \quad (5.1)$$

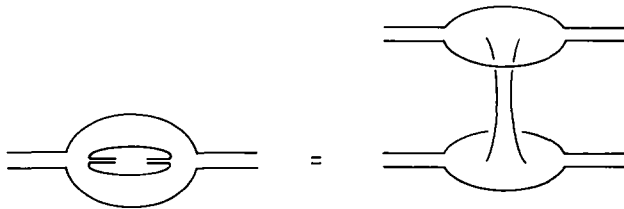


Fig. 6. An open string loop diagram which is dual to closed string (graviton) exchange.

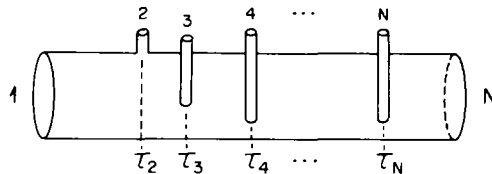


Fig. 7. World sheet for the evaluation of an N -particle tree amplitude.

where $|1\rangle$ and $\langle N|$ are particular on-shell string states, and $V(k, \tau)$ stands for one of the vertex operators constructed in sect. 2.

It is convenient to remove the τ (and σ) dependence from the vertex operators, and to absorb it in propagators. This is done by noting that the τ dependence of V is generated by the "hamiltonian"

$$H = \frac{1}{2}P^2 + 2N + 2(\tilde{N} - 1) + \sum_{I=1}^{16} (P^I)^2, \quad (5.2)$$

whereas the σ dependence can be removed by using the operator U defined in eq. (2.8). Thus any vertex can be written as

$$V(k, \tau) = e^{iH\tau} \int_0^\pi \frac{d\sigma}{\pi} U(\sigma) \tilde{V}(\tau=0, \sigma=0) U^\dagger(\sigma) e^{-iH\tau}, \quad (5.3)$$

where $\tilde{V}(\tau, \sigma)$ is the vertex operator for the emission of a pointlike string at time τ at position σ . One can now perform the integrations over the times, τ_i , and point σ_i of the string emission. Between each two vertices we then get a propagator (here we have analytically continued to imaginary $\tau = -it$)

$$\Delta = \int_0^\infty dt \int_0^\pi \frac{d\sigma}{\pi} e^{-Ht} U(\sigma), \quad (5.4)$$

where the integration is over the relative t and σ difference of adjacent vertices. Note that the integration over σ simply yields the projection operator onto states that satisfy the constraint given by eq. (2.28). Changing variables from t, σ to $z = e^{-2t + 2i\sigma}$, yields the representation

$$\Delta = \int_{|z| \leq 1} \frac{d^2z}{\pi} |z|^{1/2} P^2 + 2N - 1 + \sum_{I=1}^{16} (P^I)^2. \quad (5.5)$$

Thus the contribution to the N -point function of eq. (5.1) can be written as

$$A(1, 2, \dots, N) \sim \langle N | \tilde{V}_{N-1}(k_{N-1}) \Delta \tilde{V}_{N-2} \dots \Delta \tilde{V}_2(k_2) | 1 \rangle, \quad (5.6)$$

where we have used the fact that $H|\text{phys}\rangle = 0$, $U(\sigma)|\text{phys}\rangle = |\text{phys}\rangle$.

We must now sum over all possible τ -orderings for the emission of particles $2, \dots, N-1$ from the string. This has the effect of simply enlarging the regions of integration over the z_i variables in the propagators. To see this note that interchanging the order of two particles corresponds to changing the sign of their relative time. Under this change, $t \rightarrow -t$, $|z| \rightarrow 1/|z|$, and the interior of the unit disc, $|z| \leq 1$, is mapped onto the exterior of the unit disc, $|z| \geq 1$. Therefore if we can establish that eq. (5.1) is independent of the order of the vertex operators, the sum over all time

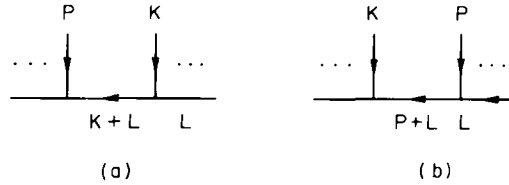


Fig. 8. Routing of internal momenta in the four-particle amplitude for two different time orderings.

orderings will be given by eq. (5.6) where the z integration in each propagator extends over the whole complex plane. In the case of the heterotic string the only new element is the left-moving piece which acts on the internal space (for external charged bosons). Consider for example the subprocess in fig. 8a. The vertex for the absorption of a boson of lattice momentum K contains the one-cocycle $C(K)$, which acts on the intermediate state of lattice momenta L and give a factor $\epsilon(K, L)$. The second vertex then yields a factor of $\epsilon(P, K + L)$. Altogether we have

$$\epsilon(P, K + L)\epsilon(K, L) = \epsilon(P, K)\epsilon(P, L)\epsilon(K, L) = (-)^{P \cdot K} \epsilon(K, P + L)\epsilon(P, L), \quad (5.7)$$

where we have used the bilinearity of the ϵ 's and the fact that $\epsilon(P, K) = (-)^{P \cdot K} \epsilon(K, P)$. The last factor in eq. (5.7) is precisely the group structure that comes from fig. 8b, times a factor $(-)^{P \cdot K}$. This phase factor compensates the one that arises when we reverse the order: $e^{2iP \cdot X}$ and $e^{2iK \cdot X}$ (as in eq. (2.57) of (1)).

We shall now illustrate the calculation of tree amplitudes in the simplest non-trivial case of the scattering amplitudes for charged, massless gauge bosons. This is easiest to evaluate, since the relevant vertex generator, eq. (2.16), involves only tachyonic matrix elements in the left-moving Fock space. The full four-point function for gauge bosons (both charged and neutral) can then be obtained from group invariance.

The amplitude for the scattering of gauge mesons of momenta k'_a , lattice momenta (weights) K'_a , and polarizations $\rho'_a(k'_a)$ (for convenience, we choose the momenta and polarization to be purely transverse, and all momenta to be outgoing so that $(k'_a)^2 = \sum_{a=1}^4 k'_a{}^2 = \sum_{a=1}^4 K'_a{}^2 = 0$), is then given by

$$A(1, \dots, 4) = g^2 \int \frac{d^2 z}{4\pi} \langle \rho_4, K_4 | \left(\rho_3 B(k_3) e^{iK_3 \cdot X} : e^{2iK'_3 \cdot X'} : C(K_3) \right) z^{N_2 \tilde{N}_2 - 1} (P')^{1/2} \\ \times |z|^{4P^2 - 2} \left(\rho_2 B(K_2) e^{iK_2 \cdot X} : e^{2iK'_2 \cdot X'} : C(K_2) \right) | \rho_1(k_1), K_1 \rangle, \quad (5.8)$$

where, for example, $|\rho_1(k_1), K_1\rangle = \rho'_1(k_1)|i\rangle_R \times |K_1\rangle_L$, and X and X' are evaluated at $\tau = \sigma = 0$.

Since the left- and right-moving operators commute the above matrix element factorizes into a product. The algebra in the left-moving sector (which involves only $\bar{z}$) is identical to that needed to calculate the scattering amplitude of bosonic string ground states, whereas the right-moving sector involves the algebra required for calculating the superstring scattering amplitude. The calculations are tedious, but straightforward (using coherent state techniques) [8, 28] and yield for the left-moving (right-moving) integrands $I(\bar{z})$ ($I(z)$)

$$I(\bar{z}) = \epsilon(K_3, -K_4) \epsilon(K_2, K_1) \bar{z}^{-\frac{1}{2}s + \frac{1}{2}t} S^{-2} (1 - \bar{z})^{-\frac{1}{2}t + \frac{1}{2}T - 2}, \quad (5.9)$$

$$\begin{aligned} I(z) = & \left(1 + \frac{1}{8}t\right) \rho_1 \cdot \rho_4 \rho_2 \cdot \rho_3 z^{-\frac{1}{2}s} (1 - z)^{-\frac{1}{2}t - 2} \\ & + \frac{1}{4} \left[-\rho_1 \cdot \rho_4 \rho_2 \cdot k_1 \rho_3 k_4 + \rho_2 \cdot \rho_4 \rho_1 \cdot k_2 \rho_3 \cdot k_4 + \rho_1 \cdot \rho_3 \rho_2 \cdot k_1 \rho_4 \cdot k_3 \right. \\ & - \rho_2 \cdot \rho_3 \rho_1 \cdot k_2 \rho_4 \cdot k_3 + \rho_1 \cdot \rho_2 (\rho_3 \cdot k_2 \rho_4 \cdot k_3 - \rho_3 \cdot k_4 \rho_4 \cdot k_4) \\ & + \rho_3 \cdot \rho_4 (\rho_1 \cdot k_2 \rho_2 \cdot k_3 - \rho_1 \cdot k_3 \rho_2 \cdot k_1) - \rho_1 \cdot \rho_2 \rho_3 \cdot \rho_4 k_2 \cdot k_3 \left. \right] z^{-\frac{1}{2}s - 1} \\ & + \frac{1}{4} \left[\rho_1 \cdot \rho_4 (\rho_2 \cdot k_3 \rho_3 \cdot k_4 + \rho_2 \cdot k_1 \rho_3 \cdot k_2 + \rho_2 \cdot k_3 \rho_3 \cdot k_2) \right. \\ & - \rho_2 \cdot \rho_4 (\rho_1 \cdot k_3 \rho_3 \cdot k_2 + \rho_1 \cdot k_2 \rho_3 \cdot k_2) \\ & + \rho_1 \cdot \rho_2 (\rho_3 \cdot k_2 \rho_4 \cdot k_2 + \rho_3 \cdot k_2 \rho_4 \cdot k_3) \\ & + \rho_3 \cdot \rho_4 (\rho_1 \cdot k_3 \rho_1 \cdot k_3 + \rho_1 \cdot k_2 \rho_2 \cdot k_3) \\ & - \rho_1 \cdot \rho_3 (\rho_2 \cdot k_3 \rho_4 \cdot k_3 + \rho_2 \cdot k_3 \rho_4 \cdot k_2) \\ & + \rho_2 \cdot \rho_3 (\rho_1 \cdot k_3 \rho_4 \cdot k_2 - \rho_1 \cdot k_4 \rho_4 \cdot k_3) + \rho_1 \cdot \rho_3 \rho_2 \cdot \rho_4 k_2 \cdot k_3 \\ & \left. - \rho_1 \cdot \rho_2 \rho_3 \cdot \rho_4 k_2 \cdot k_3 \right] z^{-\frac{1}{2}s} (1 - z)^{-\frac{1}{2}t - 1}. \end{aligned} \quad (5.10)$$

Here we have defined the usual invariants: $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$, and $u = (k_1 + k_3)^2$; $s + t + u = 0$. We also define the lattice momenta invariants

$$S = (K_1 + K_2)^2, \quad T = (K_2 + K_3)^2, \quad U = (K_1 + K_3)^2, \quad S + T + U = 8; \quad (5.11)$$

since $K_a^2 = 2$ and $\sum_{a=1}^4 K_a^I = 0$. The allowed values of S (and of course T and U) are 0 (if $K_1 = -K_2$), 2 (if $K_1 \cdot K_2 = -1$), 4 (if $K_1 \cdot K_2 = 0$), 6 (if $K_1 \cdot K_2 = 1$) and 8 (if $K_1 = K_2$).

The $z, \bar{z}$ integrals can be performed with the aid of an integration formula given in appendix A. The result then can be expressed in a remarkably simple form

$$A(1, \dots, 4) = g^2 K(\rho_1, k_1; \dots; \rho_4 k_4) \epsilon \times \frac{\Gamma(-1 - \frac{1}{8}u + \frac{1}{2}U) \Gamma(-1 - \frac{1}{8}s + \frac{1}{2}S) \Gamma(-1 - \frac{1}{8}t + \frac{1}{2}T)}{\Gamma(1 + \frac{1}{8}u) \Gamma(1 + \frac{1}{8}s) \Gamma(1 + \frac{1}{8}t)}, \quad (5.12)$$

where K is the kinematical factor familiar from the superstring

$$\begin{aligned} K = & -\frac{1}{4}(st\rho_1 \cdot \rho_3 \rho_2 \cdot \rho_4 + su\rho_2 \cdot \rho_3 \rho_1 \cdot \rho_4 + tu\rho_1 \cdot \rho_2 \rho_3 \cdot \rho_4) \\ & + \frac{1}{2}s(\rho_1 \cdot k_4 \rho_3 \cdot k_2 \rho_2 \cdot \rho_4 + \rho_2 \cdot k_3 \rho_4 \cdot k_1 \rho_1 \cdot \rho_3 + \rho_1 \cdot k_3 \rho_4 \cdot k_2 \rho_2 \cdot \rho_3 \\ & \quad + \rho_2 \cdot k_4 \rho_3 \cdot k_1 \rho_1 \cdot \rho_4) \\ & + \frac{1}{2}t(\rho_2 \cdot k_1 \rho_4 \cdot k_3 \rho_3 \cdot \rho_1 + \rho_3 \cdot k_4 \rho_1 \cdot k_2 \rho_2 \cdot \rho_4 + \rho_2 \cdot k_4 \rho_1 \cdot k_3 \rho_3 \cdot \rho_4 \\ & \quad + \rho_3 \cdot k_1 \rho_4 \cdot k_2 \rho_2 \cdot \rho_1) \\ & + \frac{1}{2}u(\rho_1 \cdot k_2 \rho_4 \cdot k_3 \rho_3 \cdot \rho_2 + \rho_3 \cdot k_4 \rho_2 \cdot k_1 \rho_1 \cdot \rho_4 + \rho_1 \cdot k_4 \rho_2 \cdot k_3 \rho_3 \cdot \rho_4 \\ & \quad + \rho_3 \cdot k_2 \rho_4 \cdot k_1 \rho_1 \cdot \rho_2) \end{aligned} \quad (5.13)$$

and ϵ is the phase factor

$$\epsilon(K_1, K_2, K_3, K_4) = \epsilon(K_3, -K_4) \epsilon(K_1, K_2) (-)^{K_1 \cdot K_3}. \quad (5.14)$$

This expression exhibits remarkable duality properties. It is invariant under interchange of the labels of any two of the gauge bosons, and contains both graviton and gauge boson exchanges.* The symmetry of the kinematical factor K and the product of Γ -functions is manifest. The invariance of ϵ follows from the properties of the two-cocycle $\epsilon(K, L)$. Since the symmetric group S_4 is generated by the permutations (12) and (1234) it suffices to show that $\epsilon(1, 2, 3, 4) = \epsilon(2, 1, 3, 4) = \epsilon(2, 3, 4, 1)$ where we have used the shorthand notation $\epsilon(K_1, K_2, K_3, K_4) = \epsilon(1, 2, 3, 4)$. This is easily established, e.g.

$$\begin{aligned} \epsilon(1, 2, 3, 4) &= \epsilon(3, -4) \epsilon(1, 2) (-)^{1 \cdot 3} = -\epsilon(3, 1) \epsilon(3, 2) \epsilon(1, 2) (-)^{1 \cdot 3} \\ & \quad [\text{Using } 4 = -1 - 2 - 3, \text{ the bilinearity of } \epsilon \text{ and } \epsilon(3, -3) = -1] \\ &= -\epsilon(3, 2) \epsilon(3, 1) \epsilon(2, 1) (-)^{1 \cdot 2 + 1 \cdot 3} \quad [\text{using eq. (2.18)}] \\ &= -\epsilon(3, 2) \epsilon(3, 1) \epsilon(2, 1) (-)^{2 \cdot 3} \quad [\text{using } 1 \cdot 2 + 1 \cdot 3 = 2 \cdot 3 \text{ mod } 2] \\ &= \epsilon(2, 1, 3, 4). \end{aligned}$$

* For example the factor $\Gamma(-1 - \frac{1}{8}s + \frac{1}{2}S)$ contains a massless pole, $s=0$, if $S=0$ corresponding to a neutral gauge boson or a graviton) or if $S=2$ (corresponding to a charged gauge boson).

Thus ϵ and hence this four-point amplitude is totally symmetric under interchange of the K 's. Note the simplicity of using the weight lattice in doing the group theory for what are rather complicated groups ($G = E_8 \times E_8$ or spin (32)/ Z_2).

The above amplitude can be recast in a more conventional form, in which the gauge mesons are labelled by matrices T^a of the gauge group $G = \text{SO}(32)$ or $E_8 \times E_8$. Using the current-algebra techniques explained in sect. 3, we rewrite the factor in the amplitude which depends on the gauge symmetry degrees of freedom as (using eq. (3.10))

$$\begin{aligned} & \langle T_4 | V(T_3) \bar{z}^{N-2} V(T_2) | T_1 \rangle \\ &= \langle 0 | T_4 \cdot Q_{-1} \left(\sum_{n=-\infty}^{\infty} T_3 \cdot Q_n \right) \bar{z}^{N-2} \left(\sum_{m=-\infty}^{\infty} T_2 \cdot Q_m \right) T_1 \cdot Q_{-1} | 0 \rangle. \end{aligned} \quad (5.15)$$

Because Q_n annihilates the incoming asymptotic vacuum state for n non-negative and annihilates the outgoing asymptotic vacuum state for n nonpositive, we can evaluate this via the current algebra relations (3.6)–(3.9), obtaining

$$\begin{aligned} I(\bar{z}) &= \bar{z}^{-\frac{1}{2}N-2} (1-\bar{z})^{-\frac{1}{2}N} \left\{ \bar{z}^2 \frac{1}{2} \text{tr}(T_1 T_2) \frac{1}{2} \text{tr}(T_3 T_4) + \frac{1}{2} \text{tr}(T_1 T_3) \frac{1}{2} \text{tr}(T_2 T_4) \right. \\ &\quad + \frac{\bar{z}^2}{(1-\bar{z})^2} \frac{1}{2} \text{tr}(T_1 T_4) \frac{1}{2} \text{tr}(T_2 T_3) + \frac{\bar{z}}{1-\bar{z}} \frac{1}{2} \text{tr}([T_1, T_2][T_3, T_4]) \\ &\quad \left. + \frac{\bar{z}^2}{1-\bar{z}} \frac{1}{2} \text{tr}([T_1, T_3][T_4, T_2]) \right\}. \end{aligned} \quad (5.16)$$

(We take here matrices T in the vector representation of $\text{SO}(32)$ or of an $\text{SO}(16) \times \text{SO}(16)$ subgroup of $E_8 \times E_8$, so that $\text{tr}(T^a T^b) = 2\delta^{ab}$; for the adjoint representation, divide each tr by 30.) Performing the integrals as before, we obtain for the scattering amplitude

$$\begin{aligned} A(1, \dots, 4) &= g^2 \frac{\Gamma(-\frac{1}{8}s) \Gamma(-\frac{1}{8}t) \Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s) \Gamma(\frac{1}{8}t) \Gamma(\frac{1}{8}u)} K(\rho_1, k_1; \dots; \rho_4, k_4) \\ &\times \left\{ \left[\frac{\frac{1}{2} \text{tr}(T_1 T_2) \frac{1}{2} \text{tr}(T_3 T_4)}{s(1+\frac{1}{8}s)} + \frac{\frac{1}{2} \text{tr}(T_1 T_4) \frac{1}{2} \text{tr}(T_2 T_3)}{t(1+\frac{1}{8}t)} + \frac{\frac{1}{2} \text{tr}(T_1 T_3) \frac{1}{2} \text{tr}(T_2 T_4)}{u(1+\frac{1}{8}u)} \right] \right. \\ &\quad \left. + \left[\frac{8}{st} \text{tr}(T_1 T_2 T_3 T_4) + \frac{8}{su} \text{tr}(T_1 T_3 T_4 T_2) + \frac{8}{tu} \text{tr}(T_1 T_4 T_2 T_3) \right] \right\}. \end{aligned} \quad (5.17)$$

We have written (5.15) in a form which exhibits in a simple fashion the low-energy limit of the tree diagrams ($s, t, u \ll 1$ (in units of $1/\alpha'$)): the product of Γ functions, which contains the massive poles, equals one in this limit. The first term in braces contains a sum of graviton exchanges in the s , t , and u channels whereas the second term consists of a sum of terms that describes gauge meson exchange. Thus the term (for $s \approx 0$) $\text{tr}(T_1 T_2) \text{tr}(T_3 T_4)/s$ contains a spin-two graviton pole in the s -channel, if this channel has singlet quantum numbers ($\text{tr}(T_1 T_2) \neq 0$). The terms $(1/s)[\text{tr}(T_1 T_2 T_3 T_4)/t + \text{tr}(T_1 T_3 T_4 T_2)/u]$ contains for $s \sim 0$ a vector meson pole $(1/2st)\text{tr}([T_1 T_2][T_3 T_4])$, with quantum numbers of the adjoint representation of G (since $[T_1, T_2]$ is in the adjoint representation). Thus we see explicitly how one string diagram (that of fig. 4a) contains all possible gauge-boson and graviton exchanges in all channels.

This tree amplitude can be used to provide a check on the relation between Newton's constant, $G_N = \kappa^2/8\pi$, and the gauge coupling g_{10} . The pole at $t=0$ receives contributions from gauge boson and graviton exchange (the dilaton and antisymmetric tensor field do not contribute). Comparing (5.15) in this limit with the standard field theory result we once more deduce that $g = g_{10} = 2\kappa$ in units where $\alpha' = \frac{1}{2}$.

6. Loop amplitudes

In this section we shall explore the properties of one-loop radiative corrections to heterotic string amplitudes. We shall explicitly evaluate a particularly simple case, that of the one-loop correction to the S -matrix amplitude for the scattering of four charged vector mesons illustrated in fig. 9. This exercise not only demonstrates the existence of a well-defined procedure for constructing the interacting heterotic string in perturbation theory, but allows us to confront two important issues. These are the finiteness of perturbation theory [29] and the possible existence of global diffeomorphism anomalies [3, 4]. We shall show explicitly that the theory is finite to one-loop

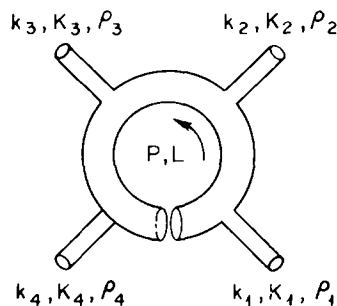


Fig. 9. Heterotic one-loop diagram for scattering of charged vector mesons.

order, explain how this confirms the vacuum stability of the theory and argue that finiteness should persist to all orders. Furthermore we shall show that the absence of global diffeomorphism anomalies requires the compactification of the sixteen internal left-moving coordinates of the heterotic string on the maximal torus of $G = E_8 \times E_8$ or spin $(32)/Z_2$.

From the earliest days of the “dual-resonance” model one worried about the existence of ultraviolet divergences [30–34]. Since the tree amplitudes contained particles of arbitrarily high spin one might have expected to find severe ultraviolet divergences in the loop amplitudes. However this was not the case. The one-loop amplitude of the open bosonic string contained only a very simple divergence, which could be absorbed by a change in the parameters of the theory [34]. This was interpreted as a sign that the dual resonance model was renormalizable, in the traditional sense of quantum field theory. Subsequent analysis [35] has shown however that these infinities do not arise from traditional ultraviolet divergences, but rather from infrared divergences associated with the existence of a massless, spin-zero particle (the dilaton) which can have a non-vanishing vacuum expectation value. Consequently the occurrence of infinities in string loops implies the instability of the perturbative vacuum state and not the non-renormalizability of the theory.

The reason this instability was often confused with coupling constant and slope renormalizability is that a shift of the vacuum expectation value of the dilaton field can be reabsorbed by a rescaling of the string tension α' [34]. This fact, by the way, explains why string theories appear to depend on a dimensionless coupling constant, $g^2(\alpha')^{-3}$, that can be arbitrarily adjusted. The reason is that the existing construction of string theories do not determine the vacuum expectation value of the dilaton [25]. In the absence of a dynamical choice of vacuum state the theory is then characterized by a free parameter which distinguishes different (equally acceptable) vacua. Eventually we expect that the choice of vacuum will be unique, in which case the theory will contain no adjustable parameters.

The ultraviolet finiteness of string theories is in accord with the remarks made in sect. 2, where we noted that string interactions are introduced in a purely topological fashion and in the first quantized formulation just modify the topology of the manifolds on which the strings propagate. It is then clear that, as long as reparametrization invariance is maintained, no new counterterms can be tolerated—there are simply no other consistent forms of string interactions. Furthermore the “coupling constant” of the string theory appears in a topological term in the string action. For example in a closed string theory an N -loop diagram (with N handles attached to a tree) is proportional to $g^{2N} = \exp[\ln(g^2)\chi]$, where $\chi = + \int \sqrt{g} R d^2\xi$ is the Euler character of the world sheet, which counts the number of handles. One would not expect such a term to be renormalized.

What is the source then of the infinities that sometimes appear in string loops? Consider the planar loop diagram for open strings depicted in fig. 10. The world sheet for this process has the topology of a disc, to which the external particles are

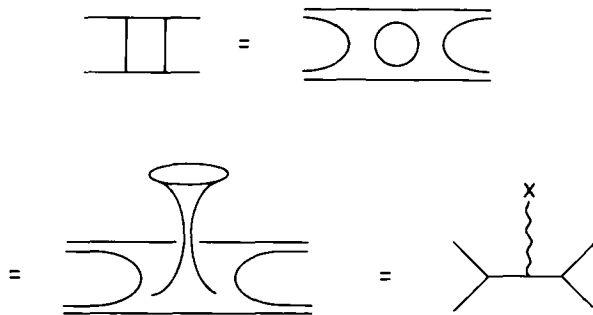


Fig. 10. One-loop diagram in an open string theory.

attached, with a hole cut out. It can be viewed as describing the rescattering of open strings (the box diagram), or equivalently as the emission of a *closed string*, which goes into the vacuum, from the tree diagram. This diagram does contain, for almost all open string theories, a divergence which arises when the diameter of the hole shrinks to zero. This region of parameter space describes, among the rest, the process in which the closed string which is emitted develops an infinitely long and infinitely narrow neck, thus corresponding to an on-mass-shell particle going into the vacuum. Since the closed string sector always contains (when constructed about flat space) a massless spin-zero particle (the “dilaton”, ϕ) one would expect a divergence to arise from this process if the dilaton tadpole is non-vanishing, since one would then be sitting on the dilaton pole. In fact one can show by explicitly factoring the loop amplitude into a tree $\times$ closed string propagator $\times$ closed string to vacuum amplitude, that the divergence arises from this source [35]. The divergence is therefore an indication of vacuum instability, $\langle \text{vac} | \phi | \text{vac} \rangle \neq 0$, and not of the existence of ultraviolet divergences in the theory. Nonetheless the instability of the vacuum in a string theory is a serious problem, since we only know how to construct these theories by perturbing about a stable vacuum state. The old open bosonic string, in fact, contained additional sources of vacuum instability, namely the existence of tachyons in the free string and the generation of a cosmological constant in one-loop order. Open superstring theories are much healthier. These have no tachyons and supersymmetry prevents a cosmological constant from developing. Green and Schwarz have argued [36] that the anomaly-free $O(32)$ superstring theory is also free of dilaton tadpole induced divergences. This is due to a cancellation of the amplitudes for the closed string dilaton to go into the vacuum by attaching to a disc (as in fig. 10) and a corresponding non-orientable diagram where it attaches to a cross cap. This cancellation is rather difficult to establish rigorously and it is not clear whether it would survive beyond one-loop order.

Closed string theories, especially supersymmetric ones, are even healthier. The old bosonic closed string theory was free of vacuum instability divergences at the tree

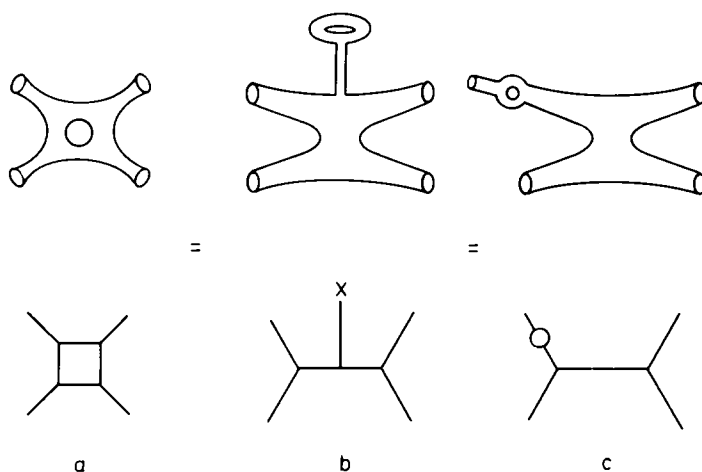


Fig. 11. One-loop diagram in a closed string theory.

level. This is because the dilaton tadpole diagram corresponds to attaching an infinitely thin cylinder to a sphere or equivalently evaluating

$$\langle \text{vac} | \phi | \text{vac} \rangle_{\text{loop}} \sim \int_{\text{sphere}} d^2 \xi \partial_\alpha X^\mu(\xi) \partial_\alpha X^\mu(\xi) e^{iS}.$$

Since the vertex operator for the dilaton, $(\partial_\alpha X^\mu)^2$, has conformal weight two, conformal invariance alone suffices to ensure (as long as it is maintained) that $\langle \phi \rangle = 0$ [15]. However at one loop order the dilaton can acquire a nonvanishing vacuum expectation value. Conformal invariance does not require that the matrix element of $(\partial X)^2$ evaluated on a torus vanish. Indeed, explicit evaluation of closed bosonic one-loop amplitudes³ shows that the only divergences that occur are due either to external particle mass renormalization (fig. 11a) or to emission of a dilaton from the closed string tree which then goes into a torus (fig. 11b). These theories are also unacceptable since they contain tachyons, and a cosmological constant will develop at the one-loop level.

Supersymmetric closed string theories, the type (II) theories and the heterotic string, are the healthiest yet. Supersymmetry requires the vanishing of the one-loop dilaton tadpole as well as the one-loop mass and vertex renormalization (for massless external particles). Thus the one-loop amplitudes, for massless external states, are completely finite. We shall see this explicitly for the heterotic string below. Furthermore we have every reason to expect that similar arguments of conformal invariance and/or supersymmetry will prevent dilaton tadpoles from developing to any order in perturbation theory. Thus we expect that the heterotic string (as well as the type (II) superstring) will prove to be finite to all orders in perturbation theory.

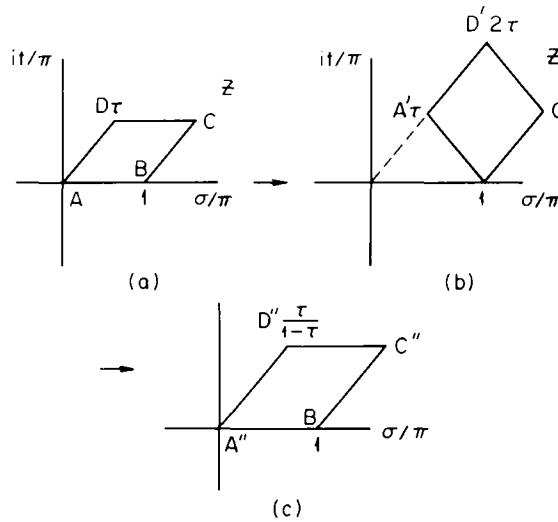


Fig. 12. The modular transformation of the torus $\tau \rightarrow \tau/(\tau - 1)$.

The second issue is that of global diffeomorphisms. The world sheet for closed string N -loop amplitudes have the topology of a sphere with N handles. In constructing these amplitudes we must sum over all such manifolds, up to reparametrizations or diffeomorphisms which are gauge symmetries of the covariant action of the string. In choosing the orthonormal and light-cone gauge conditions we have totally fixed the gauge, insofar as local diffeomorphisms are concerned. However global diffeomorphisms which cannot be reached continuously from the identity are not eliminated by the gauge choice and must be factored out by hand [3]. The group of components of (orientation preserving) diffeomorphisms is called the modular group. In the operator approach that we are following we must show that the amplitude can be written as a sum over equal contributions from manifolds which are globally diffeomorphic, and then mod out by the modular group.

Consider for example the torus, which is the relevant manifold for the one-loop amplitude. The shape of a general torus can be described by a single complex parameter conventionally termed τ (not to be confused with the timelike parameter of the string world sheet). Points on the torus may be parametrized by a complex variable $z = (\sigma + it)/\pi$ defined on the parallelogram with opposite sides identified which is depicted in fig. 12. This describes a cylinder of circumference π in the σ direction of length $\text{Im } \tau$ which is joined at the ends after a rotation of one end by $2\pi \text{ Re } \tau$. The group of global diffeomorphisms of the torus can be generated by the following discrete transformations. First, cut the torus along a circle of constant t (say $t = \pi \text{ Im } \tau$) and rotate one end by an angle 2π before rejoining. This is clearly a topologically nontrivial diffeomorphism, which cannot be reached continuously from

the identity transformation. For example, consider a curve on the torus which winds around the σ -circle (t -circle) W_σ (W_t) times. The above diffeomorphism changes the winding numbers from (W_σ, W_t) to $(W_\sigma + W_t, W_t)$. It yields an automorphism of the fundamental group, $\mathbb{Z} \times \mathbb{Z}$, of the torus. The resulting torus is described by z on a similar parallelogram, however with τ replaced by $\tau + 1$. Thus if we characterize the torus by the complex parameter τ , the above global diffeomorphism corresponds to the transformation $\tau \rightarrow \tau + 1$ of the modular group.

There are other non-trivial global diffeomorphisms of the torus. For example we can equally well cut it along a “ t -circle”, rotate by 360° and then rejoin. Consider the “ t -circle” given by $z = \alpha\tau$, $0 \leq \alpha \leq 1$. Cutting along this circle and rotating one end by $\pi|\tau|$ takes the torus (ABCD) in fig. 12a to the torus (A'BCD'). We now rescale z by $z \rightarrow z/|\tau - 1|$, so that |A'B| becomes of length one; and then rotate the parallelogram so that A' returns to the origin. In the new parallelogram (A''BC''D''), D'' is now at $\tau/(\tau - 1)$, so this global diffeomorphism corresponds to the element of the modular group $\tau \rightarrow \tau/(\tau - 1)$. Under this transformation $(W_\sigma, W_t) \rightarrow (W_\sigma, W_t + W_\sigma)$. Together with the previous transformation this generates the group of integer projective transformations, $SL(2, \mathbb{Z})$

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d = \text{integers}, \quad ad - bc = 1. \quad (6.1)$$

In particular we can generate the transformation $\tau \rightarrow -1/\tau$ which corresponds to a rotation by $\frac{1}{2}\pi$ of the parallelogram (followed by a rescaling) which interchanges σ and t ($\sigma \rightarrow t, t \rightarrow -\sigma$).

The loop amplitudes evaluated by the rules discussed above will involve an integration over the parameter τ which labels the shape of the torus. We must then verify that the integrals over τ are invariant under the transformations (6.1). If the theory passes this test of global reparametrization invariance, then we can mod out by the modular group, M , by restricting τ to lie in a fundamental region of M . This region is obtained by identifying points τ_1 and τ_2 , where τ_2 is a modular transformation of τ_1 . This is necessary in order to obtain sensible, finite one-loop amplitudes. It is this requirement of modular invariance that eliminates all but two gauge groups for the heterotic string, leaving us with $E_8 \times E_8$ or $spin(32)/\mathbb{Z}_2$.

We shall now proceed to explicitly evaluate the one-loop correction to charged vector-meson scattering (whose tree amplitude was calculated in sect. 5) again using operator methods to evaluate the vertex operator correlation function on the torus. The procedure for calculating one-loop amplitudes for external massless string states is a straightforward “sewing” together of tree amplitudes [30–32]. We simply write the amplitude for a string to emit massless particles $1, 2, \dots, N$, identifying initial and final states and sum over all possible such states. Thus, schematically, the one-loop amplitude is given by

$$A_{1 \text{ loop}}(1, 2, \dots, N) = \text{Tr}[\Delta V(N) \dots \Delta V(2) \Delta V(1)], \quad (6.2)$$

where the trace sums over all the string states, labeled by bosonic and fermionic oscillator occupation numbers, space-time momenta, and internal momenta, $V(i)$ is one of the vertex operators given in sect. 2 for the emission of a massless particle, and Δ is the propagator of eq. (5.5).

The above trace runs over both the bosonic and the fermionic Fock spaces. Thus there can be cancellations due to the relative minus sign that occurs in the odd fermion number sector. An important result of ref. [29] is that the trace over an operator in the S_0 space of the fermionic sector vanishes unless the operator contains at least eight factors of S_0^a . This follows simply from the properties of these zero mode oscillators, which satisfy eq. (2.9), from which it follows that the identity operator in the S_0 -space is

$$I = |i\rangle_R \times_R \langle i| + |a\rangle_R (\gamma^{-1})^{ba} {}_R \langle b|, \quad (6.3)$$

where $|i\rangle_R$, $|a\rangle_R$ are the right-moving ground states. Thus, for example, the trace of an operator, θ , which does not involve the S_0 's, is proportional to

$$\text{tr } I = {}_R \langle i|i \rangle_R - {}_R \langle b|(\gamma^{-1})^{ba} |a \rangle_R = \delta^{ii} - \frac{1}{2}((1 + \gamma^{11})\gamma^{-1}\gamma^{-1})^{aa} = 8 - 8 = 0.$$

Similar arguments show that at least eight S_0^a 's are required to get a nonzero result.

Consider now the one-loop amplitude for massless gravitons or gauge bosons, constructed using the vertices of eqs. (2.11), (2.14) or (2.16). Factors of S_0^a can only arise from the $\frac{1}{2}k'R''$ contribution to B' , which contains a piece $\frac{1}{16}k' \tilde{S}_0 \gamma'^{\mu\nu} S_0$. One therefore deduces that one-loop corrections to the one-, two-, or three-point functions must vanish, since they do not contain enough S_0 's. This is a typical supersymmetry non-renormalization theorem. It guarantees that the dilaton tadpole vanishes to one-loop and that the coupling g is not renormalized. The first non-trivial loop amplitude is that of the four-point function, which we proceed to calculate. We shall consider for simplicity the external particles to be charged gauge bosons, whose vertex operators are given by eq. (2.16).

The calculation proceeds by calculating the zero mode and non-zero mode contributions from the left and right-moving sectors separately and then combining the results. First we perform the trace over the S_0 right-moving space, which picks out the R_0 terms in the vertices. This yields a contribution proportional to [29]

$$\prod_{a=1}^4 \rho'^a(k_a) k'^{\mu_a} \text{tr}_{S_0} \left(\prod_a R_0'^{\mu_a} \right) = K(\rho_1 k_1, \rho_2 k_2, \rho_3 k_3, \rho_4 k_4), \quad (6.4)$$

where K is the totally symmetric kinematical factor defined in eq. (5.13). Next we trace over the remaining right-moving fermionic oscillators, the S_n 's ($n > 0$). These only appear in the propagators

$$\Delta(i) = \int \frac{d^2 z_i}{\pi} |z_i|^{4p_i^2 - 2} \bar{z}_i^{N - \tilde{N}_i - 1} \cdot \frac{1}{2} (P^i)^2, \quad (6.15)$$

where $p_i(P_i)$ is the momenta (internal momenta) of the i th internal line so that if $l(L)$ denotes the loop momenta

$$P_i = l + \sum_{j=1}^{i-1} k_j,$$

$$P_i = L + \sum_{j=1}^{i-1} K_j \equiv L + Q_i, \quad (6.6)$$

where the external particles carry momenta k_i, K_i (see fig. 9). The S_n traces yield a factor of $f(w)^8$, where we have used $\text{tr}_{S_n}[w^{1, n\bar{S} - n\gamma - S_n}] = (1 - w^n)$ and

$$f(w) = \prod_{n=1}^{\infty} (1 - w^n), \quad w = z_1 z_2 z_3 z_4. \quad (6.7)$$

$f(w)$ is just the vacuum energy on the torus of the non-zero modes of S_n .

Next we perform the traces over the bosonic oscillator modes. Using coherent state methods [31] these yield a factor of

$$T = [f(w)]^{2-d} \prod_{1 \leq i \leq j \leq 4} \prod_{m=1}^{\infty} \left[\frac{(1 - w^{m-1} C_{ji})(1 - w^m / C_{ji})}{(1 - w^m)^2} \right]^{k_i k_j / 4}, \quad (6.8)$$

where $C_{ji} = z_i \dots z_j$ and $d = 10$ for the contribution of the right-moving oscillators. The factors of $f(w)$ are again the vacuum energy of the non-zero mode oscillators. The produce on the r.h.s. is the contribution of these oscillators to the correlation function of the vertex operators.

$$\left\langle \prod_{i=1}^4 e^{i k_i X(z_i)} \right\rangle = \prod_{i < j} e^{i k_i k_j \langle X(z_i) X(z_j) \rangle}. \quad (6.9)$$

The left-moving oscillators yield a similar factor with $d = 26$, where w and C_{ji} are replaced with $\bar{w}$ and $\bar{C}_{ji}$, and $\frac{1}{4} k_i \cdot k_j$ is replaced by $\frac{1}{4} k_i \cdot k_j + K_i \cdot K_j$. The trace around the loop of the product of one cocycles, $C(K^I)$, that appear in the vertex, eq. (2.16), for the emission of charged gauge mesons yields the factor $\bar{\epsilon}(K_1, K_2, K_3, K_4) = \epsilon(K_1, K_2) \epsilon(K_3, -K_4)$. Finally the zero modes of the X 's yield an integral over the continuous ten-dimensional loop momenta, l , and a sum over all values of the lattice loop momentum L . The integral over l yields

$$M = \int d^{10}l \prod_{i=1}^4 |z_i|^{1/2} = \left(\frac{-2\pi}{\ln|w|} \right)^5 \prod_{1 \leq i < j \leq 4} \left(|C_{ji}|^{-1/2} \exp \left[\frac{\ln^2 |C_{ji}|}{2 \ln|w|} \right] \right)^{k_i \cdot k_j / 4}, \quad (6.10)$$

whereas the sum over L yields

$$\begin{aligned}\tilde{M} &= \sum_{L \in \Lambda} \prod_{i=1}^4 \bar{z}_i^{1/2} P_i \equiv \prod_{1 \leq i < j \leq 4} \left[\frac{1}{\sqrt{\bar{C}_{ij}}} \exp \left[\frac{\ln^2 \bar{C}_{ij}}{2 \ln \bar{w}} \right] \right]^{K_i K_j} \cdot \mathfrak{L}^{\mathfrak{L}} \\ \mathfrak{L} &= \sum_{L \in \Lambda} \exp \left[\frac{1}{2} \ln \bar{w} \left(L - \sum_{i=1}^4 \frac{\ln \bar{z}_i}{\ln \bar{w}} Q_i \right)^2 \right].\end{aligned}\quad (6.11)$$

with P_i given by eq. (6.6). Eqs. (6.10) and (6.11) are the contributions of the zero modes to the correlation function (6.9).

Putting all this together we obtain

$$\begin{aligned}A_{1\text{ loop}}(1, 2, 3, 4) &= \bar{e} K \int \prod_{i=1}^4 d^2 z_i \frac{1}{|w|^2} \left(\frac{-4\pi}{\ln |w|} \right)^5 \prod_{1 \leq i < j \leq 4} (\chi(C_{ij}, w))^{1/2 K_i \cdot K_j} \\ &\quad \times \bar{w}^{-1} f(\bar{w})^{-24} \prod_{1 \leq i < j \leq 4} (\psi(\bar{C}_{ij}, \bar{w}))^{K_i \cdot K_j} \cdot \mathfrak{L}^{\mathfrak{L}},\end{aligned}\quad (6.12)$$

where

$$\chi(z, w) = \exp \left[\frac{\ln^2 |z|}{2 \ln |w|} \right] \left| \frac{1-z}{\sqrt{z}} \prod_{m=1}^{\infty} \frac{(1-w^m z)(1-w^m/z)}{(1-w^m)^2} \right|, \quad (6.13)$$

$$\psi(\bar{z}, \bar{w}) = \exp \left[\frac{\ln^2 \bar{z}}{2 \ln \bar{w}} \right] \frac{1-\bar{z}}{\sqrt{\bar{z}}} \prod_{m=1}^{\infty} \frac{(1-\bar{w}^m \bar{z})(1-\bar{w}^m/\bar{z})}{(1-\bar{w}^m)^2} \quad (6.14)$$

The part of the integrand preceding the brackets is the expression that occurs for the type (II) closed superstring. Note that the integration over the phase of the z_i 's couples these factors in an intrinsically non-trivial way. It is the phase integration, of course, that projects onto intermediate states that satisfy the constraint, eq. (2.8), and ensure that the would be tachyon intermediate state is absent (see below).

Using the properties of the automorphic functions, ψ , χ and $\mathfrak{L}$ which are described in appendix B, we shall now analyze the properties of this amplitude. First we must fix the region of integration of the variables $z_1, \dots, z_4$ and $w = z_1 z_2 z_3 z_4$ which describe the points on the torus. It is convenient to use the variables

$$\begin{aligned}\nu_i &\equiv \sum_{j=1}^i \frac{\ln z_j}{2\pi i} \quad (i = 1, 2, 3), \\ \tau &= \frac{\ln w}{2\pi i}.\end{aligned}\quad (6.15)$$

Here τ is the variable used above to describe the shape of the torus. If we remember that $\ln z_i = -2[(t_i - i\sigma_i) - (t_{i-1} - i\sigma_{i-1})]$, where the i th particle is emitted at (euclidean) time t_i and position σ_i (see the discussion preceding eq. (5.5)), we would expect that the integrand is invariant under the transformations

$$\begin{aligned} \nu_i &\rightarrow \nu_i + 1, \\ \nu_i &\rightarrow \nu_i + \tau. \end{aligned} \quad (6.16)$$

The first of these corresponds to translating the position σ_i at which the i th particle is emitted by π , and the second corresponds to translating the particle in the t -direction all the way around the torus.

These invariances have been previously established for the superstring [29], so we need only verify that the remainder is likewise invariant. To see the invariance under $\nu_i \rightarrow \nu_i + \tau$ we note that under $\bar{z} \rightarrow \bar{z}\bar{w}$ one can show $\psi(\bar{z}\bar{w}, \bar{w}) = \psi(\bar{z}, \bar{w})$. Furthermore the lattice sum $\mathbb{E}$ is also invariant since the transformation $z_i \rightarrow z_i \cdot \bar{w}$ in $\sum_{L \in \Lambda} \exp[\frac{1}{2} \ln \bar{w} (L - \sum_{i=1}^4 (\ln \bar{z}_i / \ln \bar{w}) Q_i)]$, is equivalent to a shift in L by a lattice vector Q_i . To check the invariance under $\nu_i \rightarrow \nu_i + 1$ it is convenient to utilize the modular transformation property of $\mathbb{E}$, eq. (B.17), as well as eq. (B.2), to rewrite

$$\begin{aligned} \left[\prod_{1 \leq i < j \leq 4} \psi(\bar{C}_{ij}, \bar{w})^{K_i K_j} \right]_{\mathbb{E}} &= 2\pi i \left(\frac{-2\pi}{\ln \bar{w}} \right)^8 \prod_{1 \leq i < j \leq 4} \left[\frac{\theta_1(\bar{\nu}_{ij}|\bar{\tau})}{\theta_1'(0|\bar{\tau})} \right]^{K_i K_j} \\ &\times \sum_{L \in \Lambda} \exp \left[\frac{2\pi^2}{\ln \bar{w}} \left(L - \sum_{i=1}^4 \frac{Q_i \ln \bar{z}_i}{2\pi i} \right)^2 \right], \end{aligned} \quad (6.17)$$

where $\bar{\nu}_{ij} = \bar{\nu}_i - \nu_j = \ln(\bar{C}_{ij})/2\pi i$. Using the definition eq. (B.4) of θ_1 , the fact that $(\sum_{i \neq j} K_i) \cdot K_j = -K_j^2 = -2$, and the fact that a shift of $\ln \bar{z}_i$ by $2\pi i$ is equivalent to shifting the dummy variable L by a lattice vector Q_i , the invariance under $\nu_i \rightarrow \nu_i + 1$ follows.

Therefore we must restrict the region of integration of the ν 's to lie in the region

$$0 \leq \text{Im } \nu_i \leq \text{Im } \tau, \quad -\frac{1}{2} \leq \text{Re } \nu_i \leq \frac{1}{2}, \quad (6.18)$$

which is mapped by combinations of the above transformations, (6.16), onto the whole complex plane.

Finally we must determine the region of integration for the variable τ , which describes the shape of the torus. As we have discussed above, we expect the amplitude to be invariant under modular transformations, which describe the global diffeomorphisms of the torus. We recall that these are generated by $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$. Again we need only establish the modular invariance of the term in brackets in eq. (6.12), the rest is the superstring expression which has been shown to

be invariant. The invariance under $\tau \rightarrow \tau + 1$ (or $\bar{w} \rightarrow \bar{w}$) is immediate. The transformation $\tau \rightarrow -1/\tau$ ($\bar{w} \rightarrow \bar{w}' = \exp[4\pi^2/\ln \bar{w}]$, which corresponds to the reparametrization $t \leftrightarrow \sigma$, must be accompanied by $\nu \rightarrow -\nu/\tau$. Under this transformation $\psi(\bar{\nu}, \bar{\tau})$, $f(\bar{w})$ and $L(\bar{\tau}, \bar{\nu}_i)$ transform as given by eqs. (B.4), (B.17) and (B.7). Putting these all together we find

$$\begin{aligned} \frac{1}{\bar{w}} f(\bar{w})^{-24} \prod_{1 \leq i < j \leq 4} [\psi(\bar{\nu}_{ij} | \bar{\tau})]^{K_i \cdot K_j} \mathbb{L}(\tau, \nu_i) \\ \rightarrow \frac{1}{\bar{w}'} f(\bar{w}')^{-24} \cdot \bar{\tau}^{12} \left(\prod_{1 \leq i < j \leq 4} \psi \left(-\frac{\bar{\nu}_{ij}}{\bar{\tau}} \middle| -\frac{1}{\bar{\tau}} \right) \cdot \bar{\tau} \cdot e^{i\pi \bar{\nu}_{ij}^2 / \bar{\tau}} \right)^{K_i \cdot K_j} \\ \times \mathbb{L} \left(-\frac{1}{\bar{\tau}}, \frac{-\nu_i}{\bar{\tau}} \right) \cdot \left(\frac{1}{\bar{\tau}} \right)^8 \cdot \exp \left[-\frac{i\pi}{\bar{\tau}} \left(\sum_{i=1}^4 Q_i \bar{\nu}_i \right)^2 \right]. \end{aligned} \quad (6.19)$$

One then notes that

$$\prod_{1 \leq i < j \leq 4} \bar{\tau}^{K_i \cdot K_j} = \bar{\tau}^{(\sum_{i=1}^4 K_i)^2} = \bar{\tau}^{-\sum_{i=1}^4 K_i^2}$$

(since $\sum K_i = 0$) $= \bar{\tau}^{-4}$ (since $K_i^2 = 2$). The factors of $\bar{\tau}$ on the r.h.s. therefore cancel. Furthermore, using momentum conservation and the definitions of Q_i and ν_{ij} , one easily shows that $\sum_{1 \leq i < j \leq 4} \bar{\nu}_{ij}^2 K_i \cdot K_j = (\sum_{i=1}^4 Q_i \bar{\nu}_i)^2$ so that the exponential factors on the r.h.s. also cancel. This establishes the modular invariance of the heterotic amplitude for gauge groups $E_8 \times E_8$ or $\text{spin}(32)/Z_2$. For other gauge groups of rank 16 the lattice sum $\mathbb{L}$ will not transform simply and the amplitude will not be modular invariant. Physically this is due to the fact that the crucial modular transformation interchanges σ and τ ; since the internal degrees of freedom depend only on $\sigma + \tau$ the coefficients of σ (the allowed winding numbers) must coincide with the coefficients of τ (the allowed moment) i.e. the lattice must be self-dual.

Given the modular invariance of the amplitude, we must integrate τ over a fundamental region, which is mapped by the modular group onto the whole complex plane, so as to count the torus once and only once. We can thus restrict ourselves to

$$\tau \in F: \begin{cases} -\frac{1}{2} \leq \text{Re } \tau \leq \frac{1}{2}, \\ |\tau| \geq 1. \end{cases} \quad (6.20)$$

This is a particularly nice region, since it is then evident that the would be divergence arising from $w \sim 1$, $\tau \sim 0$, is absent. (If we were to take another fundamental region F' , which contains the point $\tau = 0$, we would find that the integration measure vanishes so rapidly as $\tau \rightarrow 0$ so as to render this point regular.)

Another source of infinities of the amplitude might come from the region of integration where $\bar{\nu}_{ij} \sim 0$, since here $\psi(\bar{\nu}_{ij}|\bar{\tau}) \sim \bar{\nu}_{ij}$ (see eq. (B.3)), and thus the integrand behaves as

$$\psi(\bar{\nu}_{ij}|\bar{\tau})^{K_i K_j} \sim \bar{\nu}_{ij}^{\frac{1}{2}(K_i + K_j)^2 - 2}.$$

Since $(K_i + K_j)^2$ is the square of a lattice vector, there might exist $1/\bar{\nu}_{ij}$ as well as $1/\bar{\nu}_{ij}^2$ divergences. The first of these would correspond to an intermediate one particle gauge boson state, the second to a tachyon, in the channel with momentum $(K_i + K_{i+1} + K_j)$. These singularities cannot be present since the tachyon is not in the physical Hilbert space and gauge boson one particle states cannot appear since on-shell self energy and vertex corrections vanish due to supersymmetry. In fact they are not present; this is to be expected since integrating over the *phases* of the ν_i variables eliminates these terms—recall that the source of this phase integral was the projection $N = N - 1 + \frac{1}{2}\sum (K^I)^2$ in the propagator (4.4).

Finally, the region corresponding to a potential vacuum instability (fig. 11b) is $\nu_{ij} \approx \epsilon \sim 0$, $i, j = 1 \dots 4$, but τ not necessarily zero, corresponding to a process where a four-point tree amplitude couples to a one-loop tadpole. Again the amplitude would appear to diverge, this time as ϵ^{-4} , but the integration over the phase of ϵ renders the diagram finite.

Thus we have established that, as expected, the heterotic string one loop amplitudes are finite and well behaved. In establishing this result we have made use of the modular invariance of the amplitude, which indicates the absence of global diffeomorphism anomalies. It is this requirement, in fact, that restricts the compactification of the 16 internal bosonic left-moving coordinates to lie on the maximal torus of $E_8 \times E_8$ or $\text{spin}(32)/Z_2$.

8. Discussion

In this paper we have presented the construction of the interacting heterotic string theory, shown that it is consistent and one-loop finite, and exhibited many of its properties. We constructed explicit vertex operators that describe, in light-cone gauge, the emission of massless supergravitational multiplets, or super Yang-Mills multiplets from the string, and shown that these preserve Lorentz invariance, supersymmetry and gauge invariance. The current algebraic nature of the internal symmetry group was elucidated. With the aid of the vertex operators we evaluated the tree amplitudes for the scattering of massless particles. These are used to explicitly deduce the structure of the effective field theoretic lagrangian, which governs the dynamics of the massless modes at low energies. We saw how the Chern-Simons terms, required for anomaly cancellation [22], and the dilaton couplings to R^2 and F^2 , necessary for consistent Calabi-Yau compactifications [15], emerge directly from the tree approximation.

We developed the machinery for calculating one-loop amplitudes, including the construction of automorphic functions related to the internal compactified space of the heterotic string. The requirement that there be no global diffeomorphism anomalies [4] at one-loop level provided the final element of the proof that only the gauge groups $E_8 \times E_8$ or $SO(32)$ are allowed. The finiteness of the one-loop amplitudes, expected on general grounds, was explicitly exhibited.

All in all we have brought the heterotic string to the same state of development as the older, consistent superstring theories. The few remaining gaps, the explicit demonstration that hexagon one-loop string amplitudes are nonanomalous (which has not been done yet completely for superstring theory) and the explicit construction of the light-cone gauge second quantized hamiltonian are, we believe, straightforward.

What are the prospects for the heterotic string theory? Early phenomenological investigations of the $E_8 \times E_8$ theory seem very promising. Many solutions have been discovered in which six dimensions are compactified on a Calabi-Yau manifold [15,37]. This kind of compactification works very neatly for the heterotic string, where one is forced both by requiring the absence of two-dimensional reparametrization anomalies and by demanding that the classical string equations be satisfied, to embed the spin connection (which lies in $SU(3)$) in the gauge group. This breaks $E_8 \times E_8$ to $E_6 \times E_8$, with some (topologically determined) number of chiral families in the $\underline{27}$ of E_6 . These manifolds are typically multiply connected and may contain noncontractible Wilson loops of E_8 (or E_8), which act like Higgs bosons and break E_6 down to phenomenologically acceptable low-energy groups. Moreover there exists a natural mechanism for the existence of massless weak isospin doublet Higgs bosons, without accompanying color triplets [38]. One can even envisage non-perturbative mechanisms (which are, however, familiar from field theoretic investigations) to break supersymmetry, through an E_8 gluino condensate [39]. All of this, at least to tree order, without generating a cosmological constant.

There are indications, however, that nonperturbative developments will be required in order to fully develop the phenomenology of the string. The current perturbative treatment yields no explanation as to which of the many possible vacua is the true ground state. Each of the solutions also contains many free parameters, which are analogous to “flat directions” of some effective potential, and it is not clear how these are fixed. The theory always contains at least one massless dilaton. As long as it remains massless the scale of the dilaton, and with it the magnitude of the gauge coupling, remain undetermined. It is unlikely that these can be dynamically fixed by perturbative physics. Finally, once supersymmetry is broken, the theory shows a tendency to relax back towards ten-dimensional flat Minkowski space or towards zero coupling [40].

There are two sources of non-perturbative dynamics that might play a role. One is possible non-perturbative effects of the effective strongly coupled non-linear σ -model that governs the compactification on small manifolds. The other, which is harder at

present to visualize, would employ effects in the string theory that cannot be treated by the ordinary loop expansion. Progress in either of these directions calls for deeper understanding of the underlying symmetry and dynamics of string theories. One might hope that much of the initial phenomenological success will survive these developments, and that contact will soon be made between the heterotic string and experiment.

Appendix A: A useful integral

Here we shall evaluate the integral

$$I(\alpha, n; \beta m) = \int \frac{d^2 z}{\pi} |z|^\alpha |1-z|^\beta z^n (1-z)^m, \quad (\text{A.1})$$

required to calculate heterotic tree amplitudes. I is a meromorphic function of α , β , n and m . The above integral representation converges for

$$\text{Re}(\alpha + \beta + n + m + 2) < 0, \quad \text{Re}(\alpha + n) > -2, \quad \text{Re}(\beta + m) > -2$$

and will be defined elsewhere by analytic continuation.

We first rewrite I as

$$I = \frac{1}{\Gamma(-\frac{1}{2}\alpha)\Gamma(-\frac{1}{2}\beta)} \int_0^\infty ds s^{-\frac{1}{2}\alpha-1} \int_0^\infty dt t^{-\frac{1}{2}\beta-1} \\ \times \int \frac{d^2 z}{\pi} z^n (1-z)^m e^{-s|z|^2 - t|1-z|^2}. \quad (\text{A.2})$$

The z integral is then easily evaluated by means of Wick's theorem (continued from integer $n+m$) and yields $(1/(s+t))(t/(s+t))^n (s/(s+t))^m \exp(-st/(s+t))$. It is then convenient to make a change of variables, first $t = (x/(1-x))s$ and then $u = xs$. In these variables the remaining integrals separate.

$$I = \frac{1}{\Gamma(-\frac{1}{2}\alpha)\Gamma(-\frac{1}{2}\beta)} \int_0^1 dx x^{n+\frac{1}{2}\alpha} (1-x)^{m+\frac{1}{2}\beta} \int_0^\infty du u^{-2-(\alpha+\frac{1}{2}\beta)} e^{-u} \\ = \frac{\Gamma(-1-\frac{1}{2}(\alpha+\beta))\Gamma(1+n+\frac{1}{2}\alpha)\Gamma(1+m+\frac{1}{2}\beta)}{\Gamma(-\frac{1}{2}\alpha)\Gamma(-\frac{1}{2}\beta)\Gamma(2+n+m+\frac{1}{2}(\alpha+\beta))}. \quad (\text{A.4})$$

For integer n and m , which is the case we are interested in we can rewrite I in a

more symmetrical form as

$$I(\alpha, n; \beta, m) = (-)^{n+m} \frac{\Gamma(1+n+\frac{1}{2}\alpha)\Gamma(1+m+\frac{1}{2}\beta)\Gamma(-1-n-m-\frac{1}{2}(\alpha+\beta))}{\Gamma(-\frac{1}{2}\alpha)\Gamma(-\frac{1}{2}\beta)\Gamma(2+\frac{1}{2}(\alpha+\beta))}. \quad (\text{A.5})$$

This formula can be used to evaluate all the integrals involved in the evaluation of tree amplitudes. For the case discussed in sect. 5, $n = \frac{1}{2}S$, $m = \frac{1}{2}T$ are integers and the phase factor $(-)^{n+m} = (-)^{(S+T)/2} = (-)^{K_1 \cdot K_2}$ contributes to ε in eq. (5.14).

Appendix B

In this appendix we shall discuss some of the special functions [41] that enter into the evaluation of heterotic string loop amplitudes. The oscillator traces produce the functions $\chi(z, w)$ and $\psi(\bar{z}, \bar{w})$ defined in sect. 6. These functions can be expressed in terms of Jacobi θ functions, which transform simply under modular transformations [41]. In terms of the variables

$$\nu = \frac{\ln z}{2\pi i}, \quad \tau = \frac{\ln w}{2\pi i}, \quad (\text{B.1})$$

we have

$$\psi(\bar{z}, \bar{w}) = 2\pi i e^{-i\pi\nu^2/\bar{\tau}} \frac{\theta_1(\bar{\nu}|\bar{\tau})}{\theta_1'(0|\bar{\tau})}, \quad (\text{B.2})$$

where

$$\theta_1(\bar{\nu}|\bar{\tau}) = 2f(\bar{w})(\bar{w})^{1/8} \sin \pi \bar{\nu} \prod_{n=1}^{\infty} (1 - 2\bar{w}^n \cos 2\pi \bar{\nu} + \bar{w}^{2n}) \quad (\text{B.3})$$

is a standard Jacobi θ -function. We shall need the transformation properties of θ_1 under modular transformations

$$\theta_1\left(\frac{\bar{\nu}}{c\bar{\tau}+d} \middle| \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \varepsilon \sqrt{c\bar{\tau}+d} \exp\left[\frac{-i\pi c\bar{\nu}^2}{c\bar{\tau}+d}\right] \theta_1(\bar{\nu}|\bar{\tau}), \quad (\text{B.4})$$

where $\varepsilon^8 = 1$.

The function χ , defined in eq. (6.13), can be similarly expressed as

$$\chi(z, w) = 2\pi \exp\left[\frac{-\pi(\text{Im } \nu)^2}{\text{Im } \tau}\right] \left| \frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)} \right|, \quad (\text{B.5})$$

and transforms under modular transformations as

$$\chi\left(\frac{\nu}{c\tau+d}\left|\frac{a\tau+b}{c\tau+d}\right.\right) = \frac{1}{|c\tau+d|} \chi(\nu|\tau). \quad (\text{B.6})$$

The transformation property of $f(w)$, under $\tau = \ln(w)/2\pi i \rightarrow -1/\tau = -2\pi i/\ln w'$ is

$$f(w) = \left(\frac{i}{\tau}\right)^{1/2} w^{-1/24} w'^{1/24} f(w'). \quad (\text{B.7})$$

The trace over the internal momenta yields a new function, $\mathbb{E}$, special to the heterotic string. It can be written as (for N external particles)

$$\mathbb{E}(\bar{W}, \bar{z}_i, \bar{K}_i) = \sum_{L \in \Lambda} \exp\left[\frac{1}{2} \ln \bar{w} \left(L - \sum_{i=1}^N \frac{\ln \bar{z}_i}{\ln \bar{w}} Q_i\right)^2\right],$$

$$Q_i = \sum_{j=1}^4 K_{ij} \quad (Q_i = 0). \quad (\text{B.8})$$

This function differs from the lattice sum $\tilde{M}$, defined on eq. (6.11), by the term $\prod_{i=1}^N (1/\sqrt{\bar{c}_{i\mu}}) \exp(\ln^2(\bar{c}_{i\mu})/2 \ln \bar{w})$ which has been absorbed into $\psi(\bar{c}_{i\mu}, \bar{w})$. The sum runs over all vectors L , which lie on the lattice of momenta Λ , i.e.

$$L^I = \sum_{i=1}^{16} n_i e_i^I, \quad n_i = \text{integer}, \quad (e_i^I)^2 = 2. \quad (\text{B.9})$$

We wish to establish the modular properties of this function, in particular its transformation properties under $\bar{\tau} = -\ln(\bar{w})/2\pi i \rightarrow -1/\bar{\tau} = 2\pi i/\ln \bar{w}$ (i.e. $\ln \bar{w} \rightarrow 4\pi^2/\ln \bar{w}$). This is done by a generalization of the Poisson summation formula. Let us consider the general case of the sum

$$P(X) = \sum_{L \in \Lambda} e^{\lambda(L \cdot Q \cdot X)^2}, \quad (\text{B.10})$$

where L lies on a lattice Λ of dimension d , generated by lattice vectors e_i ($i = 1, \dots, d$), and Q and X are d -dimensional vectors. Clearly P is a periodic function of X on the torus $\mathbb{R}^d/\Lambda$

$$P(X) = P(X \pm e_i). \quad (\text{B.11})$$

It therefore has a Fourier transform

$$P(X) = \sum_{M \in \Lambda^*} e^{-2\pi i M \cdot X / \text{Vol}(\Lambda)}, \quad (\text{B.12})$$

where M runs over the dual lattice, i.e. $M^I = \sum_{i=1}^d m_i e_i^{*I}$, and e_i^* generate the lattice Λ^* , dual to Λ .

$$\sum_{I=1}^d e_i^{*I} e_j^I = \delta_{ij}. \quad (\text{B.13})$$

We then evaluate $\tilde{P}(M)$ as

$$\tilde{P}(M) = \int \frac{d^d X}{\sqrt{|g|}} e^{2\pi i M \cdot X} P(X) = \sum_{L \in \Lambda} \int \frac{d^d X}{\sqrt{|g|}} e^{\lambda(X - Q + L)^2 + 2\pi i M \cdot X}, \quad (\text{B.14})$$

where g is the volume of the torus, $g = \det(e_i^I e_j^I)$. The sum over $L \in \Lambda$ extends the X integral to the whole of $\mathbb{R}^d$, yielding a simple gaussian

$$\tilde{P}(M) = \int \frac{d^d X}{\sqrt{|g|}} e^{\lambda(X - Q)^2 + 2\pi i M \cdot X} = \frac{1}{\sqrt{|g|}} e^{2\pi i M \cdot Q + \pi^2 M^2 / \lambda} \left(\frac{-\pi}{\lambda} \right)^{d/2} \quad (\text{B.15})$$

Inserting this back into (B.12), for $X=0$, yields:

$$\sum_{L \in \Lambda} e^{\lambda(L - Q)^2} = \frac{1}{\sqrt{|g|}} \left(\frac{-\pi}{\lambda} \right)^{d/2} \sum_{M \in \Lambda^*} e^{(\pi^2 / \lambda)(M - (\lambda / \pi i)Q)^2 + \lambda Q^2}. \quad (\text{B.16})$$

Therefore unless the lattice Λ is self dual, $\Lambda = \Lambda^*$, this lattice sum does not transform simply under modular transformations. As discussed in the text, this is what requires the compactification torus of the heterotic string to be R_{16}/Λ_{16} , for a self dual Λ_{16} . In that case $\Lambda = \Lambda^*$, $|g| = 1$ and $\mathfrak{Z}(\bar{w}, z_i, K_i)$ transforms simply

$$\mathfrak{Z}(\bar{w}, \bar{z}_i, K_i) = \left(\frac{-2\pi}{\ln \bar{w}} \right)^k \sum_{L \in \Lambda} \exp \left\{ \frac{2\pi^2}{\ln \bar{w}} \left(L - \sum_{i=1}^N \frac{Q_i \ln \bar{z}_i}{2\pi i} \right)^2 + \frac{1}{2 \ln \bar{w}} \left(\sum_{i=1}^N Q_i \ln \bar{z}_i \right)^2 \right\}. \quad (\text{B.17})$$

In particular if we set $Q_i = 0$, and $\ln \bar{w} = -2i\pi\tau = -2 \ln q$, then $\mathfrak{Z}$ is simply the partition function of the lattice Λ

$$\theta(q) = \sum_{L \in \Lambda} q^{L^2} = \sum_{n=0}^{\infty} q^n d(n) = \left(-\frac{\pi}{\ln q} \right)^k \theta(e^{\pi^2 / \ln q}), \quad (\text{B.18})$$

where $d(n)$ counts the number of lattice points at distance $L^2 = n$ from the origin.

References

- [1] D. Gross, J. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253
- [2] I. Frenkel and V. Kac, Invent. Math. 62 (1980) 23;
G. Segal, Comm. Math. Phys. 80 (1981) 301;
P. Goddard and D. Olive, Proc. of a Conference on Vertex operators in mathematics and physics, ed. J. Lepowsky (Springer 1985)
- [3] J. Shapiro, Phys. Rev. D5 (1972) 1945
- [4] E. Witten, Princeton preprint (1985) and Proc. of the Argonne Conference on Anomalies, geometry, and topology, ed. A. White
- [5] P. Goddard, J. Goldstone, C. Rebbi and C. Thorn, Nucl. Phys. B56 (1973) 109
- [6] S. Mandelstam, Nucl. Phys. B64 (1973) 205;
M. Kaku and K. Kikkawa, Phys. Rev. D10 (1974) 1110;
E. Cremmer and J.-L. Gervais, Nucl. Phys. B76 (1974) 209
- [7] M. Green and J. Schwarz, Nucl. Phys. B181 (1981) 502
- [8] M. Green and J. Schwarz, Nucl. Phys. B198 (1982) 252
- [9] S. Fubini and G. Veneziano, Nuovo Cim. 67A (1970) 29
- [10] E. Witten, Comm. Math. Phys. 92 (1984) 455;
A.M. Polyakov and P. Wiegman, Phys. Lett. 141B (1984) 223;
V. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83
- [11] E. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B122 (1977) 253
- [12] I.B. Frenkel, J. Lepowsky and A. Meurman, An E_8 approach to F_4 , ed. J. McKay (Springer, 1984): A moonshine module for the monster, Proc. of a Conference on Vertex operators in mathematics and physics, ed. J. Lepowsky (Springer 1984)
- [13] E. Corrigan and D. Olive, Nuovo Cim. 11A (1972) 749;
C. Thorn, Phys. Rev. D4 (1971) 1112;
J. Schwarz, Phys. Lett. 37B (1971)
- [14] D. Friedan, E. Martinec and S. Shenker, Univ. of Chicago preprint EFI 85-32
- [15] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B, to be published
- [16] C. Callan, D. Friedan, E. Martinec and M. Perry, Princeton preprint (1985);
A. Sen, Fermilab preprint (1985);
R. Rohm and E. Witten, to appear
- [17] H. Sugawara, Phys. Rev. 170 (1968) 1659
- [18] S. Coleman, D. Gross and R. Jackiw, Phys. Rev. 180 (1969) 1359;
R. Dashen and Y. Frishman, Phys. Rev. D11 (1975) 2781
- [19] J. Cohn, D. Friedan, Z. Qiu and S. Shenker, to appear
- [20] A. Neveu and J. Scherk, Nucl. Phys. B36 (1972) 155;
J. Scherk and J. Schwarz, Nucl. Phys. B81 (1974) 118;
T. Yoneya, Lett. Nuovo Cim. 8 (1973) 951;
Prog. Theor. Phys. 51 (1974) 1907
- [21] J. Schwarz, Phys. Reports 89 (1982) 223;
L. Brink, M. Green and J. Schwarz, Nucl. Phys. B198 (1982) 474
- [22] M. Green and J. Schwarz, Phys. Lett. 149B (1984) 117
- [23] X.-G. Wen and E. Witten, Princeton preprint (1985)
- [24] G. Chapline and N. Manton, Phys. Lett. 120B (1983) 105;
E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B195 (1982) 97
- [25] E. Witten, Phys. Lett. 149B (1984) 351
- [26] G. Moore and P. Nelson, Phys. Rev. Lett. 53 (1984) 1519
- [27] C. Hull, IAS preprints (1985);
L. Alvarez-Gaumé and P. Ginsparg, Harvard preprint HUTP 85/A030 (1985)
- [28] S. Fubini, D. Gordon and G. Veneziano, Phys. Lett. 29B (1969) 679;
Y. Nambu, Proc. Int. Conf. on Symmetries and quark models, Wayne State Univ., preprint (1969);
L. Susskind, Phys. Rev. D1 (1970) 1182
- [29] M. Green and J. Schwarz, Nucl. Phys. B198 (1982) 441; see also ref. [36]
- [30] K. Kikkawa, B. Sakita and M. Virasoro, Phys. Rev. 184 (1969) 1701

- [31] D. Amati, C. Bouchiat and J.-L. Gervais, *Nuovo Cim. Lett.* 2 (1969) 399
- [32] K. Bardakci, M. Halpern and J. Shapiro, *Phys. Rev.* 185 (1969) 1910
- [33] M. Kaku and C. Thorn, *Phys. Rev. D1* (1970) 2860;
D. Gross, A. Neveu, J. Scherk and J. Schwarz, *Phys. Rev. D2* (1970) 697;
G. Frye and L. Susskind, *Phys. Lett.* 31B (1970) 589
- [34] A. Neveu and J. Scherk, *Phys. Rev. D1* (1970) 2355; see also G. Frye and L. Susskind, *Phys. Lett.* 31B (1970) 537
- [35] E. Cremmer and J. Scherk, unpublished (see J. Scherk, *Rev. Mod. Phys.* 47 (1975) 123);
J. Shapiro, *Phys. Rev. D11* (1975) 2937;
M. Ademollo et al., *Nucl. Phys.* B94 (1975) 221
- [36] M. Green and J. Schwarz, *Phys. Lett.* 151B (1985) 21
- [37] S.T. Yau, talk at the Argonne Symposium on Anomalies, geometry, and topology;
A. Strominger and E. Witten, New manifolds for superstring compactification, *Comm. Math. Phys.*,
to be published
- [38] E. Witten, Symmetry breaking patterns in superstring models, Princeton preprint (1985)
- [39] M. Dine, R. Rohm, N. Seiberg and E. Witten, Gluino condensation in superstring models, IAS
preprint (1985)
- [40] M. Dine and N. Seiberg, Couplings and scales in superstring models, IAS preprint (1985)
- [41] A. Erdelyi et al., Higher transcendental functions," (McGraw Hill, NY, 1958), vol. 2
- [42] D. Gross and E. Witten, unpublished