

PATTERNS OF SYMMETRY BREAKING IN THE EXCEPTIONAL GROUPS*

Jeffrey A. HARVEY

California Institute of Technology, Pasadena, California 91125, USA

Received 1 October 1979

The symmetry breaking patterns for scalar fields in the low-lying representations of the exceptional groups are discussed. For scalar fields in the adjoint representation the theory possesses pseudo-Goldstone bosons in each case and the unbroken subgroup is uniquely determined by quantum corrections to the effective potential. The breaking of E_6 down to $SO(10) \times U(1)$ may be of interest for grand unified theories.

1. Introduction

Consider a gauge theory with gauge group G and scalar fields Φ transforming under some representation of G . We assume that the scalar fields develop a non-zero vacuum expectation value, either classically due to a negative mass squared term in the scalar potential or from radiative corrections at the one-loop level [1]. As a result G is broken down to a subgroup H and the gauge bosons corresponding to the broken generators of G acquire masses through the Higgs mechanism. In general H is determined by the transformation properties of the scalar fields and by the form of the effective potential. The patterns of symmetry breakdown for the unitary and orthogonal groups with Higgs scalars in various representations up to second-rank tensors have been determined by Li [2]. The number of independent parameters appearing in the Higgs potential depends on the number of independent polynomials of order four or less invariant under G that can be constructed out of the scalar fields. Weinberg [3] has pointed out that in some circumstances the requirement that the theory be renormalizable (i.e., of order four or less in the scalar fields) constrains the number of independent parameters to such an extent that the potential is forced to have a symmetry group $\bar{G}$ which is larger than G . As a result there are pseudo-Goldstone bosons which have zero mass in lowest order in the Higgs couplings, but which acquire masses in higher orders. Another consequence of this phenomenon is that the subgroup which will remain unbroken

* Work supported in part by the US Department of Energy under Contract No. DE-AC-03-79ER0068.

is not yet determined at lowest order so that higher corrections to the effective potential must be included. Several ingenious examples of this phenomenon involving semisimple groups or two or more scalar fields in different representations of G have been given in the literature [3–5]. With a view towards grand unified theories we will restrict ourselves to the case where G is simple and we have only one set of scalar fields transforming under an irreducible representation of G . Two examples of scalar fields in irreducible representations of simple groups where the potential has a higher symmetry are the five-dimensional representation of $SU(2)$ (the potential has an $O(5)$ symmetry), and the adjoint representation of $SU(3)$ (the potential has an $O(8)$ symmetry). We find that with scalar fields in the adjoint representations of the exceptional groups the potential always has a higher symmetry. We discuss the breaking of the exceptional groups by Higgs scalars in the adjoint representation; with this mechanism each group has a unique unbroken subgroup which we determine. Finally, we briefly consider Higgs scalars in the defining representations and the inclusion of fermions.

2. Adjoint breaking of the exceptional groups

For a gauge theory the invariance of the Lagrange density under the action of the gauge group implies the invariance of the Higgs potential and of the higher-order corrections to the potential. Hence the Higgs potential is constructed from the independent invariant polynomials of order four or less in the Higgs fields. For Higgs fields in the adjoint representation, the independent invariants of n th order are directly related to the possible n th order generalized Casimir invariants

$$I_n = \text{Tr} (\hat{X}_{\alpha_1} \hat{X}_{\alpha_2} \dots \hat{X}_{\alpha_n}) X^{\alpha_1} X^{\alpha_2} \dots X^{\alpha_n},$$

where the $\hat{X}_\alpha$ are matrix representations of the group generators X^α (the generators will be replaced by the Higgs fields in what follows). Taking the $\hat{X}^\alpha$ in the adjoint representation gives the more familiar form

$$I_n = C_{\alpha_1 \beta_1}^{\beta_2} C_{\alpha_2 \beta_2}^{\beta_3} \dots C_{\alpha_n \beta_n}^{\beta_1} X^{\alpha_1} X^{\alpha_2} \dots X^{\alpha_n},$$

where the $C_{\alpha\beta}^\gamma$ are the structure constants of the group. As an example, consider $SO(3)$. We have $C_{\alpha\beta}^\gamma = \epsilon_{\alpha\beta}^\gamma$ so that

$$I_2 = \epsilon_{ikl} \epsilon_{jlk} X_i X_j = -2(X_1^2 + X_2^2 + X_3^2),$$

while

$$\begin{aligned} I_3 &= \epsilon_{ilm} \epsilon_{jmn} \epsilon_{kln} X_i X_j X_k = \epsilon_{ijk} X_i X_j X_k \\ &= X_1[X_2, X_3] + X_2[X_3, X_1] + X_3[X_1, X_2] = X_1^2 + X_2^2 + X_3^2, \end{aligned}$$

so that I_3 is proportional to I_2 . For $SO(3)$ all the I_n for $n > 2$ are proportional to I_2 . Racah [6] has shown that a semisimple Lie group of rank l possesses l

independent invariants. In particular, he showed that the orders of the independent invariants for the simple Lie groups were

$$\begin{aligned}
 \text{SU}(l+1): I_2, I_3, \dots, I_{l+1}, \\
 \text{SO}(2l+1): I_2, I_4, \dots, I_{2l}, \\
 \text{Sp}(2l): I_2, I_4, \dots, I_{2l}, \\
 \text{SO}(2l): I_2, I_4, \dots, I_{2l-2}, I_l, \\
 \text{G}_2: I_2, I_6, \\
 \text{F}_4: I_2, I_6, I_8, I_{12}, \\
 \text{E}_6: I_2, I_5, I_6, I_8, I_9, I_{12}, \\
 \text{E}_7: I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}, \\
 \text{E}_8: I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}.
 \end{aligned}$$

Note that G_2 , F_4 , E_6 , E_7 and E_8 possess no I_3 or I_4 . Consequently, for the adjoint breaking of these groups the potentials have the larger symmetry groups $\text{O}(14)$, $\text{O}(52)$, $\text{O}(78)$, $\text{O}(133)$ and $\text{O}(248)$ respectively. For $\text{SU}(3)$ the invariants are I_2 and I_3 so that addition of a reflection symmetry $\Phi \rightarrow -\Phi$ to eliminate I_3 gives an $\text{O}(8)$ symmetry to the potential.

In order to determine the pattern of symmetry breakdown when $\bar{G}$ is non-trivial, it is necessary to consider higher-order terms in the effective potential which are not invariant under $\bar{G}$. The one-loop effective potential is given by [1]

$$V_1 = \frac{1}{64\pi^2} \left[3 \text{Tr} M_v^4 \ln \frac{M_v^2}{\mu^2} - 4 \text{Tr} M_f^4 \ln \frac{M_f^2}{\mu^2} + \text{Tr} M_s^4 \ln \frac{M_s^2}{\mu^2} \right], \quad (1)$$

where M_v , M_f , and M_s are the mass matrices in the broken theory for fields of spin 1, $\frac{1}{2}$ and 0 respectively, and μ is a renormalization mass. We may ignore the scalar one-loop term since it is invariant under $\bar{G}$. The inclusion of fermions will be discussed later. Hence, from now on we will be discussing only the first term in (1).

The adjoint representation may be defined through the action of the group on its Lie algebra

$$X \rightarrow g^{-1} X g,$$

where $g \in G$ and X is an element of the Lie algebra of G . We therefore represent the scalar fields Φ as elements of the Lie algebra of G . In particular, a theorem from group theory says that we may transform the vacuum expectation value of Φ , which we denote by Φ_v , so that it takes the form

$$\Phi_v = \sum_{i=1}^l \Psi_i H_i, \quad (2)$$

where l is the rank of G , and the H_i are the elements of the Cartan subalgebra (i.e., the commuting generators of G). For the $SU(n)$ groups there are $n - 1$ generators in the Cartan subalgebra and these generators may be represented by real diagonal traceless $n \times n$ matrices. For this case (2) is equivalent to the statement that a hermitian matrix can always be diagonalized by a unitary transformation.

The vector meson mass matrix is given by

$$\frac{1}{2}(M_v^2)_{ab}A_a^\mu A_{\mu b} = \frac{1}{2}\text{Tr } D_\mu \Phi_v D^\mu \Phi_v = \frac{1}{2}A_a^\mu A_{\mu b} \text{Tr} [\lambda^a, \Phi_v][\lambda^b, \Phi_v],$$

where the λ_i are the matrix representations of the group generators. Using the expansion (2) we see that M_v^2 has l zero eigenvalues corresponding to the l elements of the Cartan subalgebra. The non-zero elements of M_v^2 are given by

$$(M_v^2)_{\alpha\beta} = \text{Tr} \left[E_\alpha, \sum_{i=1}^l \Psi_i H_i \right] \left[E_\beta, \sum_{j=1}^l \Psi_j H_j \right] = \left(\sum_{i=1}^l \Psi_i \alpha_i \right)^2 \delta_{\alpha\beta},$$

where we have used the Cartan-Weyl basis for the Lie algebra

$$\begin{aligned} [H_i, H_j] &= 0 & [H_i, E_\alpha] &= \alpha_i E_\alpha, \\ [E_\alpha, E_\alpha] &= \alpha_i H_i, & [E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta}, \end{aligned}$$

and the normalization convention that $\text{Tr } E_\alpha E_\beta = \delta_{\alpha\beta}$. The α, β , etc., are the root vectors of the algebra, which span the l -dimensional root space of the algebra. We use subscripts to indicate the components of the root vectors and superscripts to distinguish different root vectors, thus α_j^i is the j th component of the i th root vector.

Note that a change in μ^2 in the one-loop effective potential corresponds to a change in the fourth-order $\bar{G}$ invariant term in the potential so that we may choose μ^2 as we please. We are therefore left with the task of minimizing the function

$$\tilde{V}_1(\psi) = \sum_i (\psi \cdot \alpha^i)^4 \ln (\psi \cdot \alpha^i)^2,$$

where

$$\psi \cdot \alpha^i = \sum_{j=1}^l \psi_j \alpha_j^i.$$

Having done this, the unbroken subgroup is determined by the zero eigenvalues of M_v^2 , i.e., those roots orthogonal to $\psi_{\min}$ where $\psi_{\min}$ is the value of ψ that minimizes $\tilde{V}_1(\psi)$. This subgroup will always contain an $U(1)$ factor since by definition Φ_v commutes with all the generators of the subgroup.

Consider first the case of $SU(3)$. Our approach is identical to that given by Coleman and Weinberg [1] for the five-dimensional representation of $SU(2)$. We adjust μ^2 so that

$$\tilde{V}_1(\psi) = \sum_{i=1}^3 (\psi \cdot \alpha^i)^4 [\ln (\psi \cdot \alpha^i)^2 - \frac{1}{2}].$$

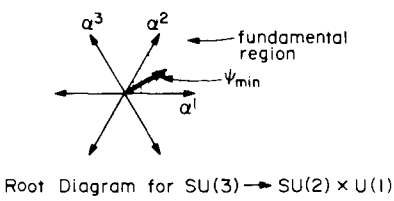


Fig. 1.

A minimum of $\tilde{V}_1(\psi)$ occurs when $(\psi \cdot \alpha^2)^2 = (\psi \cdot \alpha^1)^2 = 1$; $(\psi \cdot \alpha^3)^2 = 0$, since this corresponds to a minimum for two of the three terms in $\tilde{V}_1(\psi)$ and a maximum (with vanishing second derivative) for the third. The root diagram for $SU(3)$ given in fig. 1 shows that such a choice is possible. The non-zero roots orthogonal to $\psi_{\min}$ are $\pm \alpha^3$ so that the unbroken subgroup is $SU(2) \times U(1)$.

The root diagrams possess a symmetry group (termed the Weyl group) generated by reflections in the hyperplanes orthogonal to the roots. This partitions the root space into a number of fundamental regions ($2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ for E_8 !) which are transformed into each other under the action of the Weyl group. We may simplify the search for the minima of $\tilde{V}_1(\psi)$ by restricting the search to one fundamental region.

For each of the exceptional groups we have located the minima of $\tilde{V}_1(\psi)$ numerically. The results are given in table 1. The patterns of symmetry breakdown fall into two sets. The members of the first set are E_6 and E_7 . For these groups the symmetry breakdown is similar to that for $SU(3)$ (see fig. 1). $\psi_{\min}$ is not parallel to any of the roots and the unbroken subgroup is the maximal subgroup of maximal rank of the group which contains a $U(1)$ factor. The second set consists of G_2 , F_4 , and E_8 . For these groups $\psi_{\min}$ lies along one of the roots and the unbroken subgroup is the largest subgroup containing a $U(1)$ factor for these groups. Fig. 2 illustrates the breaking of G_2 down to $SU(2) \times U(1)$.

Table 1

Gauge group	Dimension of adjoint representation	Dimension of defining representation	Unbroken subgroup for adjoint breaking
G_2	14	7	$SU(2) \times U(1)$
F_4	52	26	$SO(7) \times U(1)$
E_6	78	27	$SO(10) \times U(1)$
E_7	133	56	$E_6 \times U(1)$
E_8	248	248	$E_7 \times U(1)$

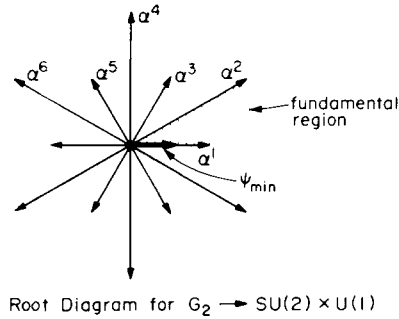


Fig. 2.

3. Defining representations of exceptional groups

We may ask whether the scalar potential possesses any larger symmetry when the scalar fields are in the defining representations of the simple groups. For the classical groups this is not the case since the classical groups may be defined as the invariance groups of the various metric forms. Cvitanović [7] has calculated the primitive invariants for the defining representations of the exceptional groups. In addition to the Kronecker tensor δ^a_b and the Levi-Civita tensor $\epsilon^{ab\dots f}$, they are

$$G_2: \delta^{ab}, f^{abc},$$

$$F_4: \delta^{ab}, d^{abc},$$

$$E_6: d^{abc},$$

$$E_7: f^{ab}, d^{abcd},$$

where the $f^{ab\dots p}$ are completely antisymmetric and the $d^{ab\dots q}$ are completely symmetric. Out of these one can construct invariants of higher order. (For example, $d^{abc} d^{cde}$ is a fourth-order invariant tensor for F_4 .) For E_8 the defining representation coincides with the adjoint representation. We see that the only group for which there is a larger symmetry is G_2 . Since G_2 lacks any symmetric invariant tensor of higher than second order, we see that if we restrict ourselves to one scalar field in the defining representation of G_2 , then the classical potential possesses an $O(7)$ symmetry. Furthermore, since there are no independent symmetric invariants of higher order, and since the effective potential must be G_2 invariant, this theory has the peculiar property that the higher symmetry is never broken.

4. Inclusion of fermions

If the fermion representation is such as to forbid coupling between the fermions and Higgs scalars to lowest order, then the fermion contribution to the one-loop

effective potential will vanish. This is the case for E_6 with the Higgs in the adjoint representation and the fermions transforming as two-component Weyl spinors in the (complex) **27** [8]. It may also be the case that all fermions occurring in the theory are much lighter than the massive vector mesons, in which case the one-loop fermion term in the effective potential is a small perturbation to the one-loop meson term and by the theory of Georgi and Glashow [9] cannot change the pattern of symmetry breaking. We have therefore ignored the effects of fermions in our calculations.

It is a pleasure to thank E. Corrigan, M. Gell-Mann and P. Ramond for discussions.

References

- [1] S. Coleman and E. Weinberg, *Phys. Rev.* D7 (1973) 1888.
- [2] L.F. Li, *Phys. Rev.* D9 (1974) 1723.
- [3] S. Weinberg, *Phys. Rev. Lett.* 29 (1972) 1698.
- [4] G. Georgi and A. Pais, *Phys. Rev.* D12 (1975) 508.
- [5] B. de Wit, S.Y. Pi and J. Smith, *Phys. Rev.* D10 (1974) 4303.
- [6] G. Racah, *Lincei Rend. Sci. Fis. Mat. Nat.* 8 (1950) 108.
- [7] P. Cvitanović, *Phys. Rev.* D14 (1976) 1536.
- [8] F. Gürsey, P. Ramond and P. Sikivie, *Phys. Lett.* B60 (1975) 177;
F. Gursey and M. Serdaroğlu, Yale report C00-3075-180 (1978);
Y. Achiman and B. Stech, *Phys. Lett.* 77B (1978) 389;
A. Shafi, $E(6)$ as a unifying gauge symmetry, Univ. of Freiburg preprint (1978).
- [9] H. Georgi and S. Glashow, *Phys. Rev.* D6 (1972) 2977.