

Magnetic monopoles in $N=4$ supersymmetric low-energy superstring theory

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A magnetic monopole solution to the field equations of $N=4$ super Yang–Mills coupled to $N=4$ supergravity in $3+1$ dimensions is presented. The solution preserves half of the supersymmetries and satisfies a Bogomol'nyi bound. The corresponding metric has no event horizons or singularities for arbitrary values of the monopole mass or charge.

1. Introduction

There is increasing evidence that there is a rich theory of semi-classical solutions to superstring theory. The classical solution corresponding to a fundamental string has properties reminiscent of solitons in supersymmetric theories [1]. String generalizations of Yang–Mills instantons have been found with [2–5] or without [6,7] Yang–Mills gauge fields and may be interpreted in ten dimensions as five-brane solitons. There are also hints of some kind of duality [1], perhaps relating strings and five-branes [3] or fundamental and solitonic strings [8]. This duality is modeled after the duality between magnetic and electric charge first discussed by Montonen and Olive [9] in the context of monopole solutions to Yang–Mills–Higgs theory. It is now understood that the duality of Montonen and Olive, if it holds at all, can only hold in $N=4$ super Yang–Mills theory since it is only there that the monopole supermultiplet coincides with the super-gauge multiplet [10].

In this paper we will construct generalized magnetic monopoles which solve the coupled $N=4$ super Yang–Mills supergravity equations of motion which arise in the low-energy approximation to string the-

ory compactified down to four dimensions on a six-torus. Our motivation is to better understand the duality conjectured in string theory as well as to have more prosaic examples of soliton solutions to string theory in $3+1$ dimensions. Magnetic monopole solutions to string theory, including BPS monopoles in theories with extended supersymmetry, have also been discussed in ref. [11].

We will first discuss the basic form of the solution and then in the following section describe the fermion zero modes of the solution and a Bogomol'nyi bound for the mass. We then briefly discuss the dyonic excitations of these solutions and end with some discussion.

2. The solution

Our starting point in constructing the $N=4$ monopole is $D=10$, $N=1$ super Yang–Mills coupled to supergravity. This theory can be dimensionally reduced to give the $N=4$ theory in $3+1$ dimensions. Equivalently, this corresponds to the massless sector after compactifying six internal dimensions of the string on a torus.

The ten-dimensional action can be written in “sigma model” variables as

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{g} e^{-2\phi} \times [R + 4(\partial\phi)^2 - \frac{1}{3}H^2 - \alpha' \text{tr} F^2 + \dots], \quad (2.1)$$

with $H = \alpha' (dB + \omega_{\frac{1}{3}}^1 - \omega_{\frac{1}{3}}^2)$. We wish to look for a solution to the resulting equations of motion with the background Fermi fields set to zero. Demanding that this solution is supersymmetric means (in ten dimensions) that there exists a positive chirality Majorana–Weyl spinor ϵ satisfying

$$\delta\chi = -\frac{1}{4g_{10}} F_{MN} \Gamma^{MN} \epsilon = 0, \quad (2.2a)$$

$$\delta\lambda = -\frac{\sqrt{2}}{4\kappa} (\Gamma^M \partial_M \phi - \frac{1}{6} H_{MNP} \Gamma^{MNP}) \epsilon = 0, \quad (2.2b)$$

$$\delta\psi_M = \frac{1}{\kappa} \mathcal{D}_M \epsilon = 0, \quad (2.2c)$$

where $\mathcal{D}_M = \nabla_M - \frac{1}{4} H_{MAB} \Gamma^{AB} = \partial_M + \frac{1}{4} (\omega_{MAB} - H_{MAB}) \times \Gamma^{AB}$ is the generalized covariant derivative. g_{10} is the ten-dimensional gauge coupling, related in the usual way to κ and α' . The precise coefficients in (2.2) will be used later. In addition, we must satisfy the Bianchi identity

$$dH = \alpha' (\text{tr} R \wedge R - \text{tr} F \wedge F), \quad (2.3)$$

where the gauge trace is in the fundamental representation (and defined by $\text{tr} F^2 = \frac{1}{30} \text{Tr} F^2$ for E_8 with Tr the trace in the adjoint representation).

In dimensionally reducing this theory to four dimensions, we choose the space–time indices to be 0, 1, 2, 3, the internal six-dimensional indices to be 4, 5, ..., 9, and require that all fields be independent of the internal space coordinates. The ten-dimensional fields can then be decomposed into their four-dimensional parts as follows. Under $\text{SO}(9,1) \supset \text{SO}(3,1) \otimes \text{SO}(6)$, the ten-dimensional vielbein breaks up into the four-dimensional vielbein, a vector with $\text{SO}(6)$ index, and a set of 21 scalars transforming as a second rank symmetric tensor under $\text{SO}(6)$. The vector gauge field A_M decomposes as $10 \rightarrow (4,1) \oplus (1,6)$ giving the four-dimensional gauge field and six scalars which take values in the adjoint of the gauge group. The ten-dimensional antisymmetric tensor field B_{MN}

decomposes as $45 \rightarrow (6,1) \oplus (4,6) \oplus (1,15)$ into the four-dimensional B field, a vector with $\text{SO}(6)$ index, and 15 scalars. Ten-dimensional Majorana–Weyl fermions break up into a set of four four-dimensional Dirac fermions according to $16 \rightarrow (2,4) \oplus (\bar{2},\bar{4})$, thus giving us $N=4$. We see that there is an overall $\text{SO}(6) \cong \text{SU}(4)$ global symmetry in the $N=4$ theory corresponding to rotations in internal space.

Although we wish to describe a solution of the four-dimensional $N=4$ theory, it is more convenient to use ten-dimensional notation. Thus in the following we will explicitly write out ten-dimensional fields. They can straightforwardly be broken up into their constituent four-dimensional fields as described above. One subtlety in this approach is that in order to have a canonical Einstein term in four dimensions, we must perform a Weyl rescaling of the space–time metric. This is all well known from the Kaluza–Klein theory.

In ref. [3], Strominger presented a generalization of the self-dual ansatz for Yang–Mills gauge fields which preserves half of the ten-dimensional supersymmetries of a given four-dimensional chirality. Using the well-known rewriting of the Bogomol’nyi equations for BPS monopoles, $B = \pm D\phi$, in terms of a dimensional reduction of the self-dual Yang–Mills equations we can equally well use this ansatz as a starting point for the construction of a supersymmetric monopole solution. To proceed, let $\mu, \nu, \dots = 1, 2, 3, 4$ denote the space indices in which the solution lives with 3+1 space–time indices given by 0, 1, 2, 3. Although we have picked a given direction in internal space, this can be related to any other one by the internal $\text{SO}(6)$ symmetry. The ansatz that preserves half of the supersymmetries of definite chirality, $\epsilon_{\pm} = \pm \Gamma^{1234} \rho_{\pm}$ (note that this is not space–time chirality), is then given by

$$F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu}{}^{\lambda\sigma} F_{\lambda\sigma}, \quad (2.4a)$$

$$H_{\mu\nu\lambda} = \mp \epsilon_{\mu\nu\lambda}{}^{\sigma} \partial_{\sigma} \phi, \quad (2.4b)$$

$$g_{MN} = \text{diag}(-1, e^{2\phi}, e^{2\phi}, e^{2\phi}, e^{2\phi}, 1, 1, 1, 1, 1), \quad (2.4c)$$

$$\nabla_{\rho} \nabla^{\rho} \phi = \mp \frac{1}{4} \alpha' \epsilon^{\mu\nu\lambda\sigma} \text{tr} F_{\mu\nu} F_{\lambda\sigma}. \quad (2.4d)$$

All other field components vanish.

The potentials for the Higgs fields $A_4, \dots, A_9$ has flat directions corresponding to configurations with $\text{tr}([A_M, A_N][A^M, A^N]) = 0$ with $M, N = 4, \dots, 9$. For

simplicity we single out the field A_4 and set $A_5=A_6=\dots=A_9=0$. Then $\mathcal{A}\equiv A_4$ has a flat potential and can take on a vacuum expectation value to become a Higgs field. From now on we will work with gauge group $SU(2)$. The extension to gauge group $SO(32)$ or $E_8\otimes E_8$ involves new bosonic and fermionic collective coordinates in a way that is well understood from standard monopole physics. The BPS solution for the gauge and Higgs fields is then

$$A_i^a = \epsilon_{iab} \frac{x^b}{r^2} (K-1), \quad \mathcal{A}^a \equiv A_4^a = \frac{x^a}{r^2} H, \quad (2.5)$$

where

$$H = Cr \coth Cr - 1, \quad K = \frac{Cr}{\sinh Cr} \quad (2.6)$$

and $C = \langle \mathcal{A} \rangle$ is the vacuum expectation value of the Higgs field. Here, $i, j, \dots = 1, 2, 3$ are space indices and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. This solution has topological charge 1 and satisfies (2.4a) with a minus sign so it is anti self-dual in this space. When we reduce to 3+1 dimensions, we demand that all fields be independent of the internal space directions. The dilaton equation, (2.4d), then becomes

$$\begin{aligned} \partial_i \partial_i e^{2\phi} &= -\alpha' \operatorname{tr} F_{\mu\nu} F_{\mu\nu} = -2\alpha' \operatorname{tr} F_{jk} F_{jk} \\ &= -4\alpha' \frac{1}{r^4} [(K^2-1)^2 + 2H^2 K^2], \end{aligned} \quad (2.7)$$

with solution

$$e^{2\phi} = e^{2\phi_0} + 2\alpha' \frac{1}{r^2} (1 - K^2 + 2H), \quad (2.8)$$

which is well behaved at the core with asymptotic behavior $e^{2\phi} \rightarrow e^{2\phi_0} + 2\alpha' C^2 + O(r^2)$ as $r \rightarrow 0$. The resulting three-form field strength is

$$H_{ij4} = 2\alpha' \epsilon_{ijk} \frac{x^k}{r^4} H(1 - K^2) = -\alpha' \omega_{ij4}^Y, \quad (2.9)$$

where ω^Y is the Yang–Mills Chern–Simons three-form. This shows that $dB=0$ everywhere, so that we can trivially set $B=0$. The solution (2.8) agrees with that given by ref. [11] to lowest order in $\alpha' C^2$.

Note that the metric is well behaved everywhere and has no singularities. However, it is not isotropic in the internal space. Viewed as a toroidal compactification from ten dimensions, we see that one of the radii is varying with a space–time dependence of

$$\tilde{e}^{2\phi(r)} \sim e^{2\phi_0} + 4\alpha' C \frac{1}{r} \quad \text{as } r \rightarrow \infty. \quad (2.10)$$

Although we have started from a single monopole solution in (2.5), we could in general have used any (anti) self-dual multi-monopole field configuration. In principle, we are still able to solve the constraint (2.4d), so exact multi-monopole configurations do exist. This is not a surprise for supersymmetric solitons and as usual is connected with a no force condition resulting from saturation of a Bogomol'nyi bound.

3. Zero modes and a Bogomol'nyi bound

It is well known that the monopole ground state must form a representation of the fermion zero mode algebra. One possible source of these zero modes comes from supersymmetry transformations. In ten dimensions, there are 16 independent Majorana–Weyl spinors ϵ_i of which eight are chiral and eight are anti-chiral in four dimensions. A supersymmetry transformation of the solution (2.4) will, if it is non-zero, give rise to a solution of the fermion equations of motion. Using these ϵ_i in the supersymmetry transformation, (2.2), we see that the eight ϵ_i of opposite chirality to the solution lead to non-vanishing values for the fermion fields. This is the manifestation of partially broken supersymmetry in the monopole background.

If we assume these eight zero modes arising from the supersymmetry are the only one present, then, as shown by Osborn in ref. [10], their quantization leads to an $N=4$ supermultiplet of one spin 1, four spin $\frac{1}{2}$, and five spin 0 states for the monopole. The addition of gravity does not affect this argument. Since this monopole supermultiplet coincides with the supergauge multiplet, this strongly suggests the presence of a generalized Montonen and Olive duality between the magnetic and electric charges [9,10].

As a check on zero mode counting, we may appeal to an index theorem argument. For a massless Dirac field, χ , in $\mathbb{R}^3$ in the presence of an $SU(2)$ monopole, the Callias index theorem predicts two zero modes for χ in the adjoint representation of $SU(2)$ [12]. Because $N=4$ fermions live in a 4 of $SU(4)$, this gives

a total of eight zero modes in agreement with the above supersymmetry argument.

Bogomol'nyi bounds in low-energy superstring theory have previously been derived in the context of superstring solitons [2] and the heterotic instanton [3]. Here we will show the existence of such a bound in the coupled $N=4$, $D=4$ super Yang–Mills supergravity theory. Such a bound can be derived from the N extended supersymmetry algebra which admits a set of central charges [13]. In the following, we extend the previous work of refs. [2,3] to the case of $N=4$, $D=4$ supersymmetry and show how the 12 possible central charges arise.

Since we wish to find a bound for the ADM mass as expressed in terms of the standard “Einstein” metric, we find it convenient to write the action in canonical variables

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{\hat{g}} \left[\hat{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{3}e^{-\phi} H^2 - \alpha' e^{-\phi/2} \text{tr} F^2 + \dots \right], \quad (3.1)$$

which is related to the “sigma model” action (2.1) through a Weyl rescaling of the metric according to $\hat{g}_{MN} = e^{-\phi/2} g_{MN}$. Since we must use the transformation properties of the physical gravitino, we also let $\hat{\psi}_M = e^{-\phi/8} (\psi_M + \frac{1}{4}\sqrt{2} \Gamma_M \lambda)$ which gives the standard supersymmetry transformation laws

$$\delta \hat{\chi} = -\frac{1}{4g_{10}} e^{-\phi/4} F_{MN} \hat{F}^{MN} \epsilon, \quad (3.2a)$$

$$\delta \hat{\lambda} = -\frac{\sqrt{2}}{4\kappa} (\hat{F}^M \partial_M \phi - \frac{1}{6} e^{-\phi/2} H_{MNP} \hat{F}^{MNP}) \epsilon, \quad (3.2b)$$

$$\begin{aligned} \delta \hat{\psi}_M &= \frac{1}{\kappa} \hat{\mathcal{D}}_M \epsilon \\ &= \frac{1}{\kappa} [\hat{\nabla}_M + \frac{1}{48} e^{-\phi/2} (\hat{F}_M \hat{F}^{NPQ} - 12 \delta_M^N \hat{F}^{PQ}) H_{NPQ}] \epsilon. \end{aligned} \quad (3.2c)$$

In deriving the Bogomol'nyi bound, we define the mass and charges of the system by the asymptotic behavior of the various fields at spatial infinity. In order to do this consistently, we require that space-time approaches flat Minkowski space at infinity and does not mix with the internal space. Our starting point is Nester's form [14] in ten dimensions written with the supercovariant derivative

$$\hat{N}^{MN} = \bar{\epsilon}' \hat{F}^{MNP} \hat{\mathcal{D}}_P \epsilon. \quad (3.3)$$

If this is the ordinary covariant derivative, then the surface integral of $\hat{N}^{MN}$ on a space-like boundary of space-time is proportional to the (ten-dimensional) ADM momentum of the system. The supercovariantization will add to this terms involving the H field.

The central charges in this theory come from the Kaluza–Klein compactification of the metric and H field to four dimensions. To make this clear, we break up the ten-dimensional fields into their four-dimensional components. Here, we let $\mu, \nu, \dots = 0, 1, 2, 3$ denote space-time indices and $i, j, \dots = 4, 5, \dots, 9$ denote internal space indices. The compactification of the vielbein yields an $SO(6)$ invariant set of $U(1)$ gauge fields with field strength $F_{(i)\mu\nu} = \partial_\mu \tilde{g}_{i\nu} - \partial_\nu \tilde{g}_{i\mu}$. Similarly, for the H field, we have a set of six $U(1)$ field strengths $H_{i\mu\nu}$. Since the supercovariant derivative in Nester's form involves the “torsion” from H along with the spin connection, we only need to take into account the linear combination of these $U(1)$ fields

$$T_{(i)\mu\nu} = F_{(i)\mu\nu} - 2 e^{-\phi/2} H_{i\mu\nu}. \quad (3.4)$$

We may integrate this field strength at spatial infinity to obtain the six “electric” charges

$$Q_i = \frac{1}{4\pi} \int_{S_\infty^2} \frac{1}{2} T_{(i)\mu\nu} d\hat{\Sigma}^{\mu\nu}. \quad (3.5)$$

In addition, there are six topological or “magnetic” charges obtained by integrating the space-time dual of the field strength, $*T_{(i)\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\lambda\sigma} T_{(i)\lambda\sigma}$

$$P_i = \frac{1}{4\pi} \int_{S_\infty^2} \frac{1}{2} *T_{(i)\mu\nu} d\hat{\Sigma}^{\mu\nu}. \quad (3.6)$$

These are the 12 central charges of this $N=4$, $D=4$ theory.

The surface integral of Nester's form can now be written as

$$\begin{aligned} \int_{\partial\Sigma} \frac{1}{2} \hat{N}^{MN} d\hat{\Sigma}_{MN} &= \int_{\partial\Sigma} \frac{1}{2} \bar{\epsilon}_0 [\hat{F}^{\mu\nu\rho} \hat{\nabla}_{4\rho} \\ &+ \frac{1}{4} (\hat{F}^\mu \hat{g}^{\nu\lambda} - \hat{F}^\nu \hat{g}^{\mu\lambda}) \partial_\lambda \log A] \epsilon_0 d\hat{\Sigma}_{\mu\nu} \\ &+ \pi V_6 \bar{\epsilon}_0 (Q_i + P_i \Gamma^5) \hat{F}^i \epsilon_0, \end{aligned} \quad (3.7)$$

where $\partial\Sigma = S_\infty^2 \times T^6$ is the two-sphere at spatial infinity.

ity times the internal space. Integrals over $\partial\Sigma$ can be broken up into space-time and internal space parts with the internal volume given by $V_6 = \int_{T^6} d\mathcal{Z}$. In the above we have assumed the spinors ϵ, ϵ' approach constant commuting spinors ϵ_0, ϵ'_0 at spatial infinity. $\hat{\nabla}$ is the four-dimensional truncation of the covariant derivative, $\hat{\nabla}_{4\rho} = \partial_\rho + \frac{1}{4}\hat{\omega}_{\rho\lambda\sigma}\hat{F}^{\lambda\sigma}$, and $\Delta = \det \hat{g}_{ij}$ so that $\partial_\lambda \log \Delta = \hat{g}^{ij} \partial_i \hat{g}_{ij}$. Γ^5 is the space-time chirality operator in flat space with eigenvalues ± 1 . In deriving (3.7), we have dropped surface integrals of fields that do not lead to conserved charges.

In order to make a connection to the $D=4$ ADM mass, we use Nester's definition

$$\bar{\epsilon}' \tilde{p}_\mu \tilde{F}^\mu \epsilon = -\frac{1}{\kappa_4^2} \int_{S_\infty^2} \frac{1}{2} \bar{\epsilon}' \tilde{F}^{\mu\nu\rho} \tilde{\nabla}_{4\rho} \epsilon \, d\mathcal{Z}_{\mu\nu}, \quad (3.8)$$

where $\tilde{p}^\mu$ is the $D=4$ ADM momentum. The above fields are defined according to the canonical $D=4$ theory with the Einstein action

$$S_4 = \int d^4x \frac{1}{2\kappa_4^2} \sqrt{\tilde{g}} \tilde{R}. \quad (3.9)$$

A simple Kaluza-Klein compactification of (3.1) shows the relation between four- and ten-dimensional quantities:

$$\kappa_4 = \frac{\kappa \Delta^{1/4}}{\sqrt{V_6}}, \quad (3.10)$$

$$\tilde{g}_{\mu\nu} = \Delta^{1/2} \hat{g}_{\mu\nu}.$$

Because of the Weyl factor, $\Delta^{1/2}$, the ADM mass expression written in ten-dimensional variables becomes

$$\begin{aligned} \bar{\epsilon}' \tilde{p}_\mu \tilde{F}^\mu \epsilon = & -\frac{1}{\kappa^2} \int_{\partial\Sigma} \frac{1}{2} \bar{\epsilon}' [\hat{F}^{\mu\nu\rho} \hat{\nabla}_{4\rho} \\ & + \frac{1}{4} (\hat{F}^\mu \hat{g}^{\nu\lambda} - \hat{F}^\nu \hat{g}^{\mu\lambda}) \partial_\lambda \log \Delta] \epsilon \, d\mathcal{Z}_{\mu\nu}. \end{aligned} \quad (3.11)$$

When combined with (3.7), this gives for the surface integral

$$\begin{aligned} \int_{\partial\Sigma} \frac{1}{2} \hat{N}^{MN} d\mathcal{Z}_{MN} \\ = -\kappa^2 \bar{\epsilon}'_0 \tilde{p}_\mu \tilde{F}^\mu \epsilon_0 + \pi V_6 \bar{\epsilon}'_0 (Q_i + P_i \Gamma^5) \hat{F}^i \epsilon_0. \end{aligned} \quad (3.12)$$

We now use the divergence theorem to write this as a volume integral

$$\int_{\partial\Sigma} \frac{1}{2} \hat{N}^{MN} d\mathcal{Z}_{MN} = \int_{\Sigma} \hat{\nabla}_M \hat{N}^{MN} d\mathcal{Z}_N, \quad (3.13)$$

where an involved calculation using considerable Dirac algebra gives for the divergence of Nester's form

$$\begin{aligned} \hat{\nabla}_M \hat{N}^{MN} = & \kappa^2 (\bar{\delta}_{\epsilon'} \hat{\psi}_M \hat{F}^{MNP} \delta_\epsilon \hat{\psi}_P + \bar{\delta}_{\epsilon'} \hat{\lambda} \hat{F}^N \delta_\epsilon \hat{\lambda} \\ & + \bar{\delta}_{\epsilon'} \hat{\chi}^a \hat{F}^N \delta_\epsilon \hat{\chi}^a) - \frac{1}{2} (\hat{G}^{NP} - \kappa^2 T^{NP}) \bar{\epsilon}' \hat{F}_P \epsilon \\ & - \frac{1}{2} e^{\phi/2} \hat{\nabla}_M (e^{-\phi} H^{MNP}) \bar{\epsilon}' \hat{F}_P \epsilon. \end{aligned} \quad (3.14)$$

Here $\hat{G}_{MN} = \hat{R}_{MN} - \frac{1}{2} \hat{g}_{MN} \hat{R}$ is the Einstein tensor and T_{MN} is the stress-energy tensor of the bosonic matter fields

$$\begin{aligned} T_{MN} = & \frac{1}{2\kappa^2} [\partial_M \phi \partial_N \phi - \frac{1}{2} \hat{g}_{MN} (\partial\phi)^2 \\ & + 2e^{-\phi} (H_{MPQ} H_N^{PQ} - \frac{1}{6} \hat{g}_{MN} H^2) \\ & + \alpha' e^{-\phi/2} (4 \operatorname{tr} F_{MP} F_N^P - \hat{g}_{MN} \operatorname{tr} F^2)], \end{aligned} \quad (3.15)$$

In calculating the divergence, the Yang-Mills gauge field only enters through the Bianchi identity for H .

The last two terms in (3.14) vanish by the Einstein and H equations of motion. We can now let $\epsilon' = \epsilon$ and impose the Witten condition $\hat{F}^i \hat{\mathcal{D}}_i \epsilon = 0$ where $i = 1, 2, \dots, 9$. This now gives the result for the volume integral

$$\begin{aligned} \int_{\Sigma} \hat{\nabla}_M \hat{N}^{MN} d\mathcal{Z}_N = & -\kappa^2 \int d^3x \sqrt{\hat{g}_{ij}} V_6 \\ & \times [\hat{g}^{ij} (\delta_\epsilon \hat{\psi}_i)^\dagger (\delta_\epsilon \hat{\psi}_j) + (\delta_\epsilon \hat{\lambda})^\dagger (\delta_\epsilon \hat{\lambda}) + (\delta_\epsilon \hat{\chi}^a)^\dagger (\delta_\epsilon \hat{\chi}^a)] \\ & \leq 0 \end{aligned} \quad (3.16)$$

for commuting spinors satisfying the Witten condition.

Equating this result to the surface integral, (3.12), we arrive at the inequality

$$\epsilon_0^\dagger \left(M - \frac{\pi}{\kappa_4^2} \hat{F}^0 (Q_i + P_i \Gamma^5) \hat{F}^i \right) \epsilon_0 \geq 0, \quad (3.17)$$

where M is the ADM mass measured in the four-dimensional theory. Since this must hold for all ϵ_0 , we finally arrive at the Bogomol'nyi bound

$$M \geq \frac{\pi}{\kappa_4^2} \sqrt{-\hat{g}^{00}} (\hat{g}^{ij} Q_i Q_j + \hat{g}^{ij} P_i P_j)^{1/2}. \quad (3.18)$$

Since (3.16) only vanishes for supersymmetric configurations, we see that this bound is saturated only when supersymmetry is at most partially broken.

The mass of the monopole solution described in the previous section can be calculated from (3.8) and turns out to be

$$M = \frac{4\pi\alpha'CV_6}{\kappa^2} e^{-\phi_0/2} = \frac{4\pi\alpha'C}{\kappa_4^2} e^{-\phi_0}. \quad (3.19)$$

All central charges vanish except for

$$P_4 = -4\alpha'C e^{-\phi_0/2}. \quad (3.20)$$

Thus the monopole solution saturates the Bogomol'nyi bound.

4. Dyons

Because the underlying ten-dimensional theory is Lorentz invariant, we may easily generalize the magnetic monopole solution. As mentioned previously, rotational invariance within the internal space gives the $N=4$ theory an overall $SO(6)$ symmetry. In addition, because we take the monopole to be time independent as well as independent of the internal space coordinates, we may rotate the monopole solution in the 0, 4 plane. Although the $D=10$ theory is invariant, the rotated solution, interpreted in four-dimensional space-time, becomes a dyon [15].

In particular, if the coordinate rotation is parametrized by $\gamma = \cosh \theta$, $\alpha = \sinh \theta$, the gauge fields become the standard dyon solution

$$A_0^a = \alpha \frac{x^a}{r^2} H, \quad A_l^a = \epsilon_{iab} \frac{x^b}{r^2} (K-1), \quad \mathcal{A}^a = \gamma \frac{x^a}{r^2} H. \quad (4.1)$$

The three-form field strength has non-vanishing components

$$H_{ij0} = \alpha H_{ij4}^0, \quad H_{ij4} = \gamma H_{ij4}^0, \quad (4.2)$$

where H_{ij4}^0 is given by (2.9). The rotated metric is no longer diagonal, but has components

$$g_{00} = \alpha^2 e^{2\phi} - \gamma^2, \quad g_{ij} = e^{2\phi} \delta_{ij}, \\ g_{44} = \gamma^2 e^{2\phi} - \alpha^2, \quad g_{MN} = \delta_{MN}, \\ g_{04} = \alpha\gamma(e^{2\phi} - 1), \quad (4.3)$$

where $i, j = 1, 2, 3$ and $M, N = 4, 5, \dots, 9$. The dilaton solution is unchanged from (2.8).

We can show that this dyon solution still preserves half of the ten-dimensional supersymmetries by repeating the analysis of section 2 in a generally covariant manner. Following ref. [16], we can introduce an orthonormal set of vectors, $\{\mathbf{m}_0, \dots, \mathbf{m}_5\}$, and a vector, $\mathbf{r}$, orthogonal to the $\mathbf{m}$'s which span the subspace 0, 4, 5, ..., 9 with $\mathbf{m}_0$ timelike. We now define the projection operator

$$P_{\pm} = \frac{1}{2} (1 \pm m_0^{N_0} m_1^{N_1} \dots m_5^{N_5} \Gamma_{N_0 N_1 \dots N_5} \Gamma^{11}), \quad (4.4)$$

which projects spinors onto a chiral four-dimensional subspace orthogonal to the space spanned by the $\mathbf{m}$'s. The dyon then satisfies the supersymmetry conditions (2.2) for anti-chiral spinors $\epsilon_- = P_- \epsilon_-$ with the choice of basis

$$\mathbf{r} = (\alpha, 0, 0, 0, -\gamma, 0, 0, 0, 0, 0), \\ \mathbf{m}_0 = (\gamma, 0, 0, 0, -\alpha, 0, 0, 0, 0, 0). \quad (4.5)$$

Since the dyon preserves this anti-chiral half of the supersymmetries, it should saturate the Bogomol'nyi bound.

In general, when $e^{2\phi_0} \neq 1$, the metric is not asymptotically flat Minkowski at spatial infinity. However, for $e^{2\phi_0} = 1$, $g_{04} \rightarrow 0$ and the Bogomol'nyi bound analysis of the previous section holds. The H field carries the dyon's "magnetic" central charge

$$P_4 = -\gamma 4\alpha' C, \quad (4.6)$$

and the metric $U(1)$ fields carry the "electric" central charge

$$Q_4 = \alpha\gamma 4\alpha' C. \quad (4.7)$$

The dyon mass is

$$M = \gamma^2 \frac{4\pi\alpha'C}{\kappa_4^2}, \quad (4.8)$$

and indeed saturates the Bogomol'nyi bound (3.18).

The treatment above has been purely classical. A correct quantum treatment of the dyon spectrum should lead to a quantized tower of electrically charged dyon states as for conventional magnetic monopoles.

5. Discussion

Magnetic monopole solutions of non-abelian Yang-

Mills–Higgs theory coupled to standard Einstein gravity have the property that at large enough values of the mass they become black holes ^{#1}. Intuitively this is due to the fact that as the Higgs vacuum expectation value C is varied, the mass M and inverse radius $1/R$ of the solution are both proportional to C so that for large enough C , we have $GM/R > 1$ indicating that the solution should be a black hole. On the other hand, the solution we have presented here has a metric without singularities or event horizons for all values of C . This appears to be related to the form of the dilaton in the solution which weakens the singularity which would be expected in pure gravity. Of course the fact that extremal supersymmetric solutions in theories of the type described here are better behaved than non-supersymmetric solutions is well known [18].

The solutions presented here are supersymmetric solutions to the low-energy string field theory equations of motion. We expect, however, that they can be perturbed to give exact solutions to string theory as long as $\alpha' C^2 \ll 1$ as was argued in ref. [5] for gauge five-branes. In this limit we also expect the general features of the solution, including the absence of any singularity or event horizon, to be shared by the full string solution. In ref. [5] it was also shown that there exist exact five-brane solutions to string theory without any perturbative corrections in α' . We have not been able to construct the magnetic monopole analogs of these solutions.

Finally, one of our main reasons for the study of these solutions was to learn more about the relation between the duality conjecture of Montonen and Olive and that of ref. [3]. In this regard it would be interesting to study the interactions of these monopoles by generalizing the results of ref. [19] to include the gravitational contribution to the metric on the multi-monopole moduli space and the effects of the fermion zero modes.

^{#1} See ref. [17] for a general discussion of monopoles coupled to gravity.

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