

Chiral Symmetry Breaking and Intersecting D-brane Systems

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These lectures discuss a holographic approach to chiral symmetry breaking utilizing intersecting D -brane systems. Two models are discussed, one which at weak coupling is related to the $3 + 1$ -dimensional Nambu–Jona-Lasinio model, and another which is related to the $1 + 1$ -dimensional Gross–Neveu model. Both models exhibit chiral symmetry breaking at strong coupling.

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1. Introduction

The general theme of these lectures is the use of the tools of string theory, in particular D -branes [1] and holography [2], to study chiral symmetry breaking in the strong interactions. Recall that QCD with N_f flavors of massless quarks with left and right-handed components q_L, q_R has a symmetry group $U(N_f)_L \times U(N_f)_R$ which acts as independent unitary rotations of q_L, q_R . We know from the identification of pions as Nambu–Goldstone bosons that this symmetry is spontaneously broken:

$$U(N_f)_L \times U(N_f)_R \rightarrow U(N_f)_V . \quad (1)$$

where $U(N_f)_V$ is generated by the diagonal combination of the generators of $U(N_f)_L$ and $U(N_f)_R$. Actually, the $U(1)$ axial symmetry is anomalous, and as a result does not lead to an additional Nambu–Goldstone boson [3]. This effect disappears in the large N_c limit which will be discussed here, and so we will ignore the subtlety in what follows.

In these lectures we will discuss D -brane configurations which exhibit the same sort of chiral symmetry breaking pattern as the strong interactions. At weak coupling we will encounter a low-energy description of these systems which involves a four-Fermion interaction of the schematic form

$$\mathcal{L}_{\text{int}} \sim (\psi_L^\dagger \psi_R)(\psi_R^\dagger \psi_L) . \quad (2)$$

The flavor and color structure will be specified later, but for now we note that this interaction

preserves a manifest $U(1)_L \times U(1)_R$ symmetry which acts as

$$\begin{aligned} U(1)_L : \quad & \psi_L \rightarrow e^{i\alpha} \psi_L, & \psi_R & \rightarrow \psi_R \\ U(1)_R : \quad & \psi_L \rightarrow \psi_L, & \psi_R & \rightarrow e^{i\beta} \psi_R. \end{aligned} \quad (3)$$

This symmetry is broken to a diagonal, vectorial $U(1)$ symmetry,

$$U(1)_V : \quad \psi_L \rightarrow e^{i\alpha} \psi_L, \quad \psi_R \rightarrow e^{i\alpha} \psi_R \quad (4)$$

if the fermion bilinear $\psi_L^\dagger \psi_R$ has a non-zero vacuum expectation value, $\langle \psi_L^\dagger \psi_R \rangle \neq 0$.

In $3 + 1$ dimensions the Lagrangian

$$\mathcal{L}_{NJL} = \mathcal{L}_{\text{free}} + G\mathcal{L}_{\text{int}} \quad (5)$$

was first introduced by Nambu and Jona-Lasinio [4] to study chiral symmetry breaking in the strong interactions. The coupling constant G has dimensions of inverse mass squared, so the theory is non-renormalizable and requires the specification of a UV cutoff Λ . The NJL model is thought to lead to chiral symmetry breaking for $G\Lambda^2 \geq 1$. In spite of its non-renormalizability, this model has been widely used in phenomenological studies of QCD at intermediate energies.

The $1 + 1$ -dimensional Lagrangian

$$\mathcal{L}_{GN} = \mathcal{L}_{\text{free}} + g^2 \mathcal{L}_{\text{int}} \quad (6)$$

describes a $1 + 1$ -dimensional version of the NJL model with a dimensionless coupling constant g^2 . This theory was studied by Gross and Neveu [5] who showed that the coupling is asymptotically free and the theory exhibits chiral symmetry breaking at large N where N is the number of fermions. Much is now known about this model, including the exact spectrum of massive states and the S -matrices for these states [6–9], and the structure of the non-trivial CFT which the model flows to in the far infrared [10,11].

In these lectures I will describe an embedding of these theories, or variants thereof, into string theory. By doing so we potentially learn something about QCD (by embedding the NJL model into a theory with good UV behavior) and have a rich laboratory for exploring the structure of holography. Of particular interest is the question

of whether or not the theories are in the same phase with broken chiral symmetry at both weak and strong coupling.

2. The D -Brane Configuration

The general set-up we will use to study chiral symmetry breaking is as follows. We take N_c Dq -branes, which we refer to as color-branes, and N_f Dp -branes and N_f $\overline{Dp}$ -branes which we call flavor branes. The color branes intersect the flavor branes on an $r + 1$ -dimensional submanifold, $Dp \cap Dq = I_r$. The flavor Dp -branes are separated from the flavor $\overline{Dp}$ -branes by a distance L along one of their transverse directions. We take $N_c \rightarrow \infty$ with N_f fixed. At least formally this allows us to ignore the coupling of the flavor branes to gravity and their backreaction on the metric and dilaton. At the next order in a $1/N_c$ expansion there are many interesting issues which deserve to be explored more carefully, but we will have our hands full just trying to understand the leading behavior of these systems at large N_c .

Before we analyze models with specific values of p, q, r it will be useful to discuss the structure of zero-modes which live at the intersections of the color branes with the flavor branes.

2.1. Intersecting Brane Zero-Modes

Let's start by considering the intersection $Dp \cap Dq = I_r$, we will then add in the intersection of the $\overline{Dp}$ -branes with the Dq -branes.

It is useful for the purposes of classification to imagine that we compactify all spatial dimensions on a torus so that we can apply T-duality as we please. In particular, we can always apply T-ualities until one of the D -branes is a $D9$ -brane. We then have, recalling the discussion in [12]

Brane Intersection	$\#ND$	E_{NS}
$D9 \cap D1 = I_1$	8	$\frac{1}{2}$
$D9 \cap D3 = I_3$	6	$\frac{1}{4}$
$D9 \cap D5 = I_5$	4	0
$D9 \cap D7 = I_7$	2	$-\frac{1}{4}$

(7)

where E_{NS} is the vacuum energy in the Neveu-Schwarz sector and $\#ND$ is the number of Neumann-Dirichlet boundary conditions. The $D9 \cap D1$ and $D9 \cap D5$ intersections preserve half of the supersymmetry of the $D9$ -brane. The $D9 \cap D7$ intersection breaks supersymmetry and because of the negative vacuum energy in the Neveu-Schwarz sector there is a tachyon in the $9 - 7$ open string sector. The $D9 \cap D3$ intersection breaks supersymmetry, but there is no tachyon. We will discuss intersections which are T-dual to the three cases without tachyons. For the most part we will focus on intersections T-dual to either $D9 \cap D3$ or $D9 \cap D1$, both of which have fermion zero-modes localized at the intersection, but no scalars or tachyons.

Let me illustrate this classification with a couple of often studied examples. Start with $D9 \cap D5$ with the $D5$ world-volume spanning the $(0, 1, 2, 3, 8, 9)$ directions and apply T-duality along the $(8, 9)$ directions. This gives $D7 \cap D3 = I_3$ with the $D3$ spanning the directions $(0, 1, 2, 3)$ and the $D7$ spanning $(0, 1, 2, 3, 4, 5, 6, 7)$. A further T-duality along the (4) direction gives $D6 \cap D4 = I_3$. In this example we take there to be N_c $D3$ or $D4$ -branes and N_f $D7$ or $D6$ -branes. The intersection in both cases is $1+1$ -dimensional and contains “quarks” transforming as $(N_c, \overline{N}_f)$ under the $U(N_c) \times U(N_f)$ gauge symmetry of the intersecting branes. If we take N_c large with N_f fixed then we can replace the $D3$ or $D4$ -branes with their near-horizon geometry and study a holographical dual to gauge theory with fundamental matter.

However, the quarks in the examples above are not chiral: the intersection has localized fermions of both chiralities, and the left and right-handed fermions have the same gauge quantum numbers. There is an easy way to understand this which I will illustrate with the $D6 \cap D4 = I_3$ example. The $(8, 9)$ directions are transverse to both the $D6$ and $D4$ -branes, so in a generic configuration the $D6$ and $D4$ will be separated by some non-zero distance L in the $(8, 9)$ directions. The $6 - 4$ open strings which give rise to the quarks will then have a mass proportional to L . Since the quarks are generically massive, and separating the D -branes does not break any of the gauge

symmetry, the quarks at the intersection cannot be chiral. This is a generic phenomenon: a non-transversal intersection leads to non-chiral quarks. We can also understand this from T-duality. The $D6 \cap D4 = I_3$ system is T-dual to $D9 \cap D5 = I_5$, carrying out the T-duality is equivalent to doing dimensional reduction from 6 to 4 spacetime dimensions, and dimensional reduction of this form leads to non-chiral fermions.

In spite of this, one can study a kind of chiral symmetry breaking in this model. The fermions are not chiral under $U(N_c) \times U(N_f)$, but they are chiral under the $SO(2)_{89}$ group of rotations on the two mutually transverse directions. This chiral symmetry is unbroken when the separation between the $D6$ and $D4$ vanishes, and it can be shown that this symmetry is dynamically broken [14–16].

If we want to instead study the dynamical breaking of the $U(N_f)_L \times U(N_f)_R$ chiral symmetry of QCD with N_f flavors of massless quarks then we must find a different starting point. The next section describes intersecting D -brane systems with precisely this symmetry structure.

2.2. Brane intersections with $U(N_f)_L \times U(N_f)_R$ symmetry.

We want to describe a field theory which has a $U(N_f)_L \times U(N_f)_R$ global symmetry in the limit of vanishing quark masses. According to the AdS/CFT correspondence, the Noether currents of this global symmetry should be dual to $U(N_f)_L \times U(N_f)_R$ gauge fields in the bulk. This, along with the fermion zero mode structure we have discussed, clearly suggests that we need two sets of N_f flavor branes with chiral fermions at the intersection of the flavor and color branes.

For example, we can start with the $D9 - D3$ system which has chiral fermions. If we start with the $D3$ world-volume spanning the $(0, 1, 2, 3)$ directions and then apply T-duality along x_4 we end up with $D8 \cap D4 = I_3$ with the $D8$ spanning $(0, 1, 2, 3, 5, 6, 7, 8, 9)$, the $D4$ spanning $(0, 1, 2, 3, 4)$. The $3 + 1$ -dimensional intersection has left-handed chiral fermion zero modes transforming as $(\overline{N}_f, N_c)$ under $U(N_f) \times U(N_c)$. We want to add to this right-handed chiral fermions transforming under a $U(N_f)_R$ gauge

group. Adding another $D8 - D4$ intersection will not accomplish this, since we have seen that the fermions at this intersection are left-handed. However, at an $\overline{D8} - D4$ intersection there is a change in the GSO projection which results in right-handed fermion zero modes. If the $D8$ and $\overline{D8}$ are coincident there will of course be a $8 - \overline{8}$ tachyon which will destabilize the system. While this starting point may be viable if one can reliably identify the ground state of the tachyon, we will, following [17], pursue a simpler approach where the $D8$ and $\overline{D8}$ are separated in the x_4 direction by a distance L which is sufficiently large that the $8 - \overline{8}$ open strings have a positive mass squared. If we also compactify the x_4 direction, $x_4 \simeq x_4 + 2\pi R_4$ we have precisely the model first introduced by Sakai and Sugimoto [17]. To summarize, the model consists of N_c color $D4$ -branes which span $(0, 1, 2, 3, 4)$, N_f flavor $D8$ and $\overline{D8}$ branes which span $(0, 1, 2, 3, 5, 6, 7, 8, 9)$ and are separated by a distance L in x_4 .

If we compactify the transverse direction x_4 on an S^1 of radius R_4 so that x_4 is identified with $x_4 + 2\pi R_4$ we can also take the $D4$ world-volume fermions to be antiperiodic in x_4 , $\psi(x_4 + 2\pi R_4) = -\psi(x_4)$. In the absence of the $D8$ -branes this is the model first considered in [21]. The antiperiodic boundary conditions break supersymmetry by giving a mass to the gauginos. There will also be radiatively induced masses for scalars, so the massless spectrum of excitation on the $D4$ -branes consists of pure Yang-Mills gauge theory with gauge group $U(N_c)$. When we add in the $D8$ and $\overline{D8}$ -branes, the chiral fermions at their intersection with the $D4$ -branes adds to the spectrum massless quarks with $U(N_f)_L \times U(N_f)_R$ chiral symmetry, precisely as in QCD in the limit of vanishing bare quark masses.

This theory has an interesting and complicated structure which has been explored in [17–19]. Some significant insights can be gained by analyzing the theory in various limits, including the limit $R_4 \rightarrow \infty$ which is what we will mainly consider in these lectures. Before doing this though I want to introduce one more model with a similar structure.

Suppose we want to study chiral symmetry “breaking” in $1 + 1$ dimensions. I put breaking in

quotes because of the well-known fact that continuous symmetries cannot be spontaneously broken in $1 + 1$ dimensions. We will ignore this subtlety for now and come back later and discuss its implications. Following our earlier logic we start with a transversal intersection of two D -branes which is T-dual to the 91 system. In particular, if we start with the 91 system with N_f $D9$ -branes and the N_c $D1$ -branes spanning $(0, 1)$. Applying T-duality along the $2, 3, 4$ directions we obtain $D6 \cap D4 = I_1$. We then add in N_f $\overline{D6}$ -branes separated a distance L along the $(2, 3, 4)$ directions from the $D6$ -brane. The resulting system has left- and right-handed chiral fermions in $1 + 1$ dimensions which transform as $(\overline{N}_f, 1, N_c) \oplus (1, \overline{N}_f, N_c)$ under the $U(N_f)_L \times U(N_f)_R \times U(N_c)$ gauge symmetry.

3. Chiral Symmetry Breaking.

I now want to start studying the dynamics of these theories, and in particular the question of whether or not the chiral symmetry is spontaneously broken. The gauge theory on the $D4$ -brane is characterized by a coupling constant with dimensions of length:

$$\lambda = \frac{g_5^2}{4\pi} N_c = g_s \ell_s N_c \quad (8)$$

where g_5 is the five-dimensional gauge coupling, g_s is the string couplings, and ℓ_s is the string length. This theory is thus strongly coupled in the UV and weakly coupled in the IR. The leading interaction between the left- and right-handed quarks at the two intersections is due to exchange of five-dimensional gauge bosons and is characterized by the dimensionless coupling λ/L . When $\lambda/L \ll 1$ we can hope to study the dynamics using a perturbative field-theory analysis. On the other hand, when $\lambda/L \gg 1$ the gauge theory is strongly coupled and our only hope is to use AdS/CFT ideas to find a dual description of the strongly coupled theory.

3.1. Weak Coupling.

The Lagrangian describing the five-dimensional gauge fields and their interaction with the quarks

q_L, q_R is given by

$$\mathcal{S} = \int d^5x \left[-\frac{1}{4g_5^2} F_{MN}^2 + \delta(x^4 + L/2) q_L^\dagger \bar{\sigma}^\mu (i\partial_\mu + A_\mu) q_L \right. \\ \left. + \delta(x^4 - L/2) q_R^\dagger \sigma^\mu (i\partial_\mu + A_\mu) q_R \right].$$

We now consider the $D4-D8-\overline{D8}$ model. To derive an effective action involving only the quarks we integrate out the five-dimensional gauge field A_M . Keeping only the quadratic terms in the gauge field Lagrangian, and working in Feynman gauge, we find the following effective action for q_L :

$$\mathcal{S}_{\text{eff}}^{48\bar{8}} = i \int d^4x q_L^\dagger \bar{\sigma}^\mu \partial_\mu q_L + S_{\text{int}} \quad (9)$$

with

$$S_{\text{int}} = -\frac{g_5^2}{4\pi} \int d^4x d^4y G(x-y, L) \left[q_L^\dagger(x) \cdot q_R(y) \right] \times \\ \left[q_R^\dagger(y) \cdot q_L(x) \right] \quad (10)$$

where $G(x^\mu, x^4)$ is proportional to the scalar propagator in $4+1$ dimensions,

$$G(x^\mu, x^4) = \frac{1}{((x^4)^2 - x_\mu x^\mu)^{\frac{3}{2}}}. \quad (11)$$

The color indices in Eq.(10) are contracted in each term in brackets separately, while the flavor ones are contracted between the terms in separate brackets. Thus we have the transformation properties

$$\left[q_L^\dagger(x) \cdot q_R(y) \right] \sim (1, N_f, \bar{N}_f) \\ \left[q_R^\dagger(y) \cdot q_L(x) \right] \sim (1, \bar{N}_f, N_f) \quad (12)$$

under $U(N_c) \times U(N_f)_L \times U(N_f)_R$.

To derive Eq.(10) we used the Fierz identities

$$(\sigma^\mu)_{\bar{\alpha}\alpha} (\sigma_\mu)_{\bar{\beta}\beta} = 2\epsilon_{\bar{\alpha}\bar{\beta}} \epsilon_{\alpha\beta} \\ (\bar{\sigma}^\mu)_{\alpha\bar{\alpha}} (\sigma_\mu)_{\bar{\beta}\beta} = 2\delta_{\alpha\beta} \delta_{\bar{\alpha}\bar{\beta}} \quad (13)$$

The theory described by Eq.(9) is the Nambu-Jona-Lasinio (NJL) model [4], except that the local interaction of the usual NJL model has been replaced with a specific non-local interaction.

We can apply the same kind of analysis to our $D4-D6-\overline{D6}$ set-up. This gives rise to an effective action which has the kinetic terms for q_L, q_R in $1+1$ dimensions and an interaction term given by

$$S_{\text{int}}^{46\bar{6}} = +\frac{g_5^2}{4\pi^2} \int d^2x d^2y G_5(x-y, L) \\ \times [q_L^\dagger(x) \cdot q_R(y)] \times [q_R^\dagger(y) q_L(x)] \quad (14)$$

This action describes a non-local version of the Gross-Neveu or Thirring model.

3.2. Range of interaction

Although in both these models we get a non-local four-fermion interaction, there is a crucial difference between these two models in addition to the obvious difference that one describes $3+1$ -dimensional physics and the other $1+1$ -dimensional physics. This has to do with the range of the non-local four-fermion interaction. In the $D4-D6-\overline{D6}$ model the integral

$$\int d^2z G_5(z, L) = \frac{2\pi}{L} \quad (15)$$

is finite. This implies that at distances much greater than L , or energies much less than $1/L$, the model becomes effectively local and we can replace $G_5(x-y, L)$ by $(2\pi/L)\delta^2(x-y)$. We will call this class of models short-range models. On the other hand, for the $D4-D8-\overline{D8}$ model the corresponding integral is infrared divergent. If we cut the integral off at a length scale z_{max} we find

$$\int d^4z G_5(z, L) \sim z_{\text{max}}. \quad (16)$$

In this case there is no energy scale below which the interaction can be treated as local. This makes it much more difficult to analyze the dynamics. We will refer to such models as long-range.

We can consider a general D -brane model of this type with color Dp -branes intersecting flavor Dq and $\overline{Dq}$ -branes on $r+1$ -dimensional intersections I_r . Then the interaction between the fermions localized at the intersections is short range if $p-r > 2$.

4. Analysis of the $D4 - D6 - \overline{D6}$ model.

It is useful to start with an analysis of chiral symmetry breaking in the short-range $D4 - D6 - \overline{D6}$ model. We will see that this theory can be analyzed reliably at both weak and strong coupling and that chiral symmetry is broken in both regimes.

4.1. Weak Coupling.

As discussed in the previous section, for the $D4 - D6 - \overline{D6}$ -model the five-dimensional Green function can be thought of as a δ -function smeared over a region of size L . Using the fact that $\int d^2x G(x, L) = 2\pi/L$ we can, at length scales large compared to L , replace the effective action S_{eff}^{466} by the local action

$$S_{\text{local}}^{466} = \int d^2x [iq_L^\dagger \bar{\sigma}^\mu \partial_\mu q_L + iq_R^\dagger \sigma^\mu \partial_\mu q_R + \frac{1}{N_c} \frac{2\pi\lambda}{L} (q_L^\dagger(x) \cdot q_R(x)) (q_R^\dagger(x) \cdot q_L(x))] \quad (17)$$

with a UV cutoff $\Lambda \simeq \frac{1}{L}$. Comparing to [5] we see that the Gross-Neveu coupling λ_{gn} is given in terms of our parameters by

$$\lambda_{gn} = \frac{2\pi\lambda}{L}. \quad (18)$$

In [5] it was found that at large N_c the model dynamically generates a fermion mass which goes like

$$M_F \simeq \mu e^{-\frac{2\pi}{\lambda_{gn}}} \quad (19)$$

where μ is an arbitrary renormalization scale and λ_{gn} the coupling at that scale. Since our model reduces in a particular limit (which will be made more precise below) to the Gross-Neveu model, it is natural to expect that it exhibits similar behavior. We next verify this by a more detailed analysis as described in [20].

To solve the theory for large N_c it is convenient to introduce a bilocal field $T(x, y)$ which is a singlet of global $U(N_c)$ and transforms as $(N_f, \bar{N}_f)$ under the $U(N_f)_L \times U(N_f)_R$. The action can be rewritten as

$$S_{\text{eff}}^{466} = i \int d^2x (q_L^\dagger \bar{\sigma}^\mu \partial_\mu q_L + q_R^\dagger \sigma^\mu \partial_\mu q_R)$$

$$+ \int d^2x d^2y \left[-\frac{N_c}{\lambda} \frac{T(x, y) \bar{T}(y, x)}{G(x - y, L)} + \bar{T}(y, x) q_L^\dagger(x) \cdot q_R(y) + T(x, y) q_R^\dagger(y) \cdot q_L(x) \right]. \quad (20)$$

Integrating out $T(x, y)$ using its equation of motion,

$$T(x, y) = \frac{\lambda}{N_c} G(x - y, L) q_L^\dagger(x) \cdot q_R(y), \quad (21)$$

one recovers the original action.

To solve the large N_c theory one instead integrates out the fermions and obtains an effective action for T , which becomes classical in the large N_c limit. Using translation invariance of the vacuum, it is enough to compute this effective action for $T(x, y) = T(|x - y|)$. Dividing by N_c and the volume of spacetime we get the effective potential

$$V_{\text{eff}} = \frac{1}{\lambda} \int d^2x \frac{T(x) \bar{T}(x)}{G(x, L)} - \int \frac{d^2k}{(2\pi)^2} \ln \left(1 + \frac{T(k) \bar{T}(k)}{k^2} \right) \quad (22)$$

where $T(k)$ is the Fourier transform of $T(x)$. Varying with respect to $\bar{T}(k)$ leads to the gap equation

$$\frac{1}{\lambda} \int d^2x \frac{T(x)}{G(x, L)} e^{-ik \cdot x} = \frac{T(k)}{k^2 + T(k) \bar{T}(k)}. \quad (23)$$

The solution of the gap equation gives the expectation value of the chiral condensate $\langle q_L^\dagger(x) \cdot q_R(y) \rangle$. This equation has a trivial solution $T = 0$, but we will next see that there is a non-trivial solution as well.

To solve the gap equation for small λ/L it is convenient to define the field $f(x, y)$ by

$$T(x, y) = \frac{L}{2\pi} G(x - y, L) f(x, y). \quad (24)$$

We then see that the expectation value of $f(x, y)$ is proportional to the condensate $\langle q_L^\dagger(x) \cdot q_R(y) \rangle$,

$$f(x - y) \equiv \langle f(x, y) \rangle = \frac{\lambda_{gn}}{N_c} \langle q_L^\dagger(x) \cdot q_R(y) \rangle. \quad (25)$$

In particular, the condensate at coincident points, $\langle q_L^\dagger(x) \cdot q_R(x) \rangle$ is proportional to $f(0)$, which we will denote by m_f . We will determine m_f as

part of the solution of the gap equation. We will assume (and justify later) that $f(x)$ is approximately constant on the scale L . Since the function $G(x, L)$ goes rapidly to zero for $x > L$ (and can be thought of as a smeared δ -function) this implies that

$$T(k) = \frac{L}{2\pi} \int d^2x G(x, L) f(x) e^{-ik \cdot x} \simeq m_f e^{-kL}. \quad (26)$$

Substituting into the gap equation gives the Fourier transform of $f(x)$:

$$\tilde{f}(k) = \lambda_{gn} \frac{m_f e^{-|k|L}}{k^2 + m_f^2 e^{-2|k|L}}. \quad (27)$$

We can now determine the mass parameter m_f by evaluating $f(0)$ in terms of its Fourier transform:

$$m_f = f(0) = \lambda_{gn} \int \frac{d^2k}{(2\pi)^2} \frac{m_f e^{-|k|L}}{k^2 + m_f^2 e^{-2|k|L}}. \quad (28)$$

Dividing by m_f and estimating the momentum integral for $m_f L \ll 1$ we find

$$1 \simeq \frac{\lambda_{gn}}{2\pi} \log(\Lambda/m_f) \quad (29)$$

where $\Lambda \sim 1/L$ is the UV cutoff of the effective GN model. Note that the dynamically generated mass scale

$$m_f \simeq \Lambda e^{-2\pi/\lambda_{gn}} \quad (30)$$

is much smaller than the cutoff Λ for small GN coupling.

To complete the discussion, we need to show that the resulting $f(x)$ is slowly varying on the scale L (to justify the approximations). Fourier transforming we find the following behavior. For $x \ll L$,

$$f(x) \simeq f(0) = m_f. \quad (31)$$

For $L \ll x \ll 1/m_f$,

$$f(x) \simeq m_f \frac{\ln(xm_f)}{\ln(Lm_f)}, \quad (32)$$

while for $x \gg 1/m_f$, f goes exponentially to zero.

Although $f(x)$ for $x > L$ is not constant, it is very slowly varying at weak coupling. Indeed,

$$\frac{f(0) - f(x)|_{x \gg L}}{f(0)} \simeq 1 - \frac{\ln(xm_f)}{\ln(Lm_f)} \simeq \frac{\lambda_{gn}}{2\pi} \ln \frac{x}{L}. \quad (33)$$

The logarithmic variation of $f(x)$ becomes important only on scales for which $\frac{\lambda_{gn}}{2\pi} \ln \frac{x}{L} \simeq 1$, that is, for $x \simeq 1/m_f$. We conclude that $f(x)$ varies on the scale $1/m_f \gg L$.

The same conclusion can be reached by examining the momentum space expression for $\tilde{f}(k)$. The Fourier transform $\tilde{f}(k)$ contains two momentum scales, m_f and $1/L$. However, most of its support is on the scale m_f . At $k = 1/L$ it has been reduced by a factor $e^{-2L/\lambda}$ from its value at the origin. This means that the scale of variation of $f(x)$ is $1/m_f$.

The solution for f gives the following position dependent condensate:

$$\langle q_L^\dagger(x) \cdot q_R(0) \rangle = N_c m_f \int_{|k| < \Lambda} \frac{d^2k}{(2\pi)^2} \frac{e^{ik \cdot x}}{k^2 + m_f^2}. \quad (34)$$

This is nothing but the massive propagator for a fermion of mass m_f . Thus, we conclude that the intersecting D-brane model generates dynamically a mass m_f for the fermions living at the intersections of the color and flavor branes. This mass is non-perturbative in the Gross-Neveu coupling and exhibits the same dependence on the coupling as in the field theoretic analysis of [5].

One can take a decoupling limit in which the physics of the brane configuration reduces precisely to the Gross-Neveu model. To do that one takes $\lambda/L \rightarrow 0$ and focuses on energy scales of order m_f . This sends the ratio of the mass of the fermions m_f and the UV cutoff scale $1/L$ to zero, and leads to a model that is precisely equivalent to that of [5]. The model with finite λ/L can be thought of as a version of the Gross-Neveu model with a finite UV cutoff.

In the above discussion we took the renormalization scale of the theory to be the UV cutoff $\Lambda \sim 1/L$. λ_{gn} is the coupling at that scale. One can choose some other renormalization scale $\mu < \Lambda$ and define the coupling λ_{gn} at that scale. The renormalization group guarantees that Green functions expressed in terms of the physical fermion mass are insensitive to such changes.

One issue that was mentioned earlier but then ignored in the analysis was the fact that continuous symmetries cannot really be broken in $1+1$ dimensions. On the other hand, the analysis through the effective potential seems to indicate a

non-zero expectation value of an order parameter that breaks $U(N_f)_L \times U(N_f)_R$ down to the diagonal subgroup $U(N_f)_V$ and fluctuations about this vacuum should lead to N_f^2 massless scalars. The resolution of this puzzle was provided in [10] where it was shown that the symmetry is restored at finite N_c . At large but finite N_c the fermion four-point correlation function behaves as

$$\langle q_L^\dagger(x) q_R(x) q_L^\dagger(y) q_R(y) \rangle \sim |x - y|^{-1/N_c} \quad (35)$$

The vanishing as $|x - y| \rightarrow \infty$ shows that the symmetry is restored at sufficiently large distances, at least for finite N_c .

It is known however that the model does contain N_f^2 massless bosons, although these are not Nambu-Goldstone bosons of a spontaneously broken symmetry. They govern the IR behavior of the model which is described by a $SU(N_f)_{N_c} \times U(1)$ WZW model.

4.2. Strong Coupling.

To analyse the dynamics at strong coupling we replace the $D4$ -branes by their near horizon geometry, described by the metric

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} (\eta_{\mu\nu} dx^\mu dx^\nu - (dx^4)^2) - \left(\frac{U}{R}\right)^{-3/2} (dU^2 + U^2 d\Omega_4^2) \quad (36)$$

and dilaton

$$e^\Phi = g_s \left(\frac{U}{R}\right)^{3/4} \quad (37)$$

where U, Ω_4 label the radial and angular directions in $(5, 6, 7, 8, 9)$ and $\mu = 0, 1, 2, 3$. The parameter R is given by

$$R^3 = \pi g_s N_c = \frac{g_5^2}{4\pi} N_c = \pi \lambda \quad (38)$$

A $D6$ -brane propagating in this geometry is described by the DBI action. Ignoring the gauge fields for now, this is just

$$S_{D6} = -T_6 \int dx^7 e^{-\Phi} \sqrt{-\det g_{MN}} \quad (39)$$

To understand what form the $D6$ -brane world-volume should take, recall that we start with

a configuration at $g_s = 0$ with the $D4$ spanning $(0, 1, 2, 3, 4)$ and $D6 - \overline{D6}$ -branes which span $(0, 1, 5, 6, 7, 8, 9)$. This configuration preserves a $SO(1, 1) \times SO(3) \times SO(5)$ symmetry. If we assume that this symmetry is preserved at strong coupling, then the $D6$ world-volume will wrap $R^{1,1} \times S^4$ and be described by a curve $U(x^4)$ in the (U, x^4) -plane. For such a configuration the DBI action reduces to

$$S_{D6} = -T_6 V_{1+1} V_4 \int dx^4 U^{5/2} \times \sqrt{1 + (R/U)^3 (U')^2} \quad (40)$$

with $U' = dU/dx^4$. The equations of motion which follow from this action have two types of solution. The first is a straight solution with constant x^4 —as at weak coupling. To describe the full configuration we add together a $D6$ solution with $x_4 = -L/2$ with a $\overline{D6}$ solution with $x^4 = L/2$. The second, curved, solution is labelled by U_0 , the minimum value of U , and is given by

$$x^4(U) = \frac{1}{5} \frac{R^{3/2}}{U_0^{1/2}} [B(3/5, 1/20) - B(U_0^5/U^5; 3/5, 1/3)] \quad (41)$$

where

$$B(z; a, b) = \int_0^z u^{a-1} (1-u)^{b-1} du \quad (42)$$

is the incomplete Beta function. Note that this solution describes both the $D6$ and $\overline{D6}$ solutions. Or rather, it describes a single $D6$ -brane which at large U looks like a separate $D6$ and $\overline{D6}$ (recall that these differ by having opposite orientation) while at small U the branes join together into a single $D6$ -brane.

Evaluating $x^4(\infty) = L/2$ we find that U_0 is related to the asymptotic separation between the $D8$ and $\overline{D8}$ -branes by

$$L = \frac{2}{5} \frac{R^{3/2}}{U_0^{1/2}} B(3/5, 1/2) \quad (43)$$

Note that this makes intuitive sense: As L increases the strength of the interaction between the quarks of opposite chirality decreases and the scale U_0 of chiral symmetry breaking decreases.

However, to determine whether chiral symmetry is really broken we need to determine which solution has lower energy. We can do this by evaluating the energy difference

$$\begin{aligned} \Delta E &= E_{\text{st}} - E_{\text{curved}} \\ &\propto \int_0^\infty U dU - \int_{U_0}^\infty U(1 - U_0^5/U^5)^{-1/2} dU \end{aligned} \quad (44)$$

Each term E_{st} and E_{curved} is of course infinite, but we can compare the energy density for each slice dU to obtain a finite answer:

$$\begin{aligned} \Delta E &= \int_0^{U_0} U dU - \int_{U_0}^\infty U(1 - U_0^5/U^5)^{-1/2} dU \\ &\simeq 0.139 U_0^2 > 0 \end{aligned}$$

so that the curved solution has lower energy.

As noted originally in [17] this also provides a beautiful geometrical picture of chiral symmetry breaking. In the UV of the field theory, which is $U \rightarrow \infty$ in the supergravity description, we see separate $D6$ and $\overline{D6}$ -branes and hence separate $U(N_f)_L$ and $U(N_f)_R$ gauge symmetries. This is the dual manifestation of the fact that chiral symmetry is restored in the UV behavior of the field theory. As we go to the IR (small U) we see that the $D6$ and $\overline{D6}$ -branes are joined together and have only a single, diagonal $U(N_f)_V$ symmetry, reflecting the breaking of chiral symmetry down to the diagonal, vectorial subgroup in the field theory.

5. Analysis of the $D4 - D8 - \overline{D8}$ system.

We now turn to an analysis of the more complicated and more realistic $D4 - D8 - \overline{D8}$ model of Sakai and Sugimoto.

5.1. Strong Coupling.

At strong coupling the analysis is very similar to what we have already done for the $D4 - D6 - \overline{D6}$ model. In order to preserve the same symmetry as at weak coupling we take the $D8$ -brane to wrap $R^{3,1} \times S^4$ and a curve $U(x^4)$ in the (U, x^4) plane. The action for a $D8$ -brane in the $D4$ -background then reduces to

$$S_{D8} = -T_8 V_{3+1} V_4 \int dx^4 U^4 \sqrt{1 + (R/U)^3 (U')^2} \quad (46)$$

Again there is a straight solution and a curved solution given by

$$\begin{aligned} x^4(U) &= \frac{1}{8} \frac{R^{3/2}}{U_0^{1/2}} [B(9/16, 1/2) \\ &\quad - B(U_0^8/U^8; 9/16, 1/2)] \end{aligned} \quad (47)$$

and again one finds that the curved solution has lower energy so that chiral symmetry is broken.

5.2. Weak Coupling

At weak coupling we can compute follow the same procedure we did at weak coupling for the $D4 - D6 - \overline{D6}$ system. We introduce a bilocal field $T(x, y)$, compute the one-loop contribution to the effective potential for a translationally invariant field, and derive a gap equation by setting $\delta V_{\text{eff}}/\delta T = 0$. This leads to the equation

$$\frac{1}{\lambda} \int d^4x \frac{T(x)}{G_5(x, L)} e^{-ik \cdot x} = \frac{T(k)}{k^2 + T(k)\overline{T}(k)} \quad (48)$$

At small x one can linearize the equation and find an approximate solution with $T(x)$ proportional to $G_5(x, L)$. However, when we try to match this solution smoothly onto a solution at large x the analysis breaks down. There are various ways to see this, but they all involve the fact that the interaction is long-range and does not turn off at any distance scale.

What then is the physics at weak coupling? The answer is not known, but if there is a way to analyze the problem it seems likely that it requires some reordering of perturbation theory in a way that would take care of the leading long-range part of the interaction. Thus the question of whether or not this model breaks chiral symmetry at both weak and strong coupling remains open.

6. Open Questions and Future Direction

There are a number of interesting directions for future research suggested by these D -brane systems exhibiting chiral symmetry breaking. These include the following.

6.1. Validity of the NJL model

The NJL model is widely used in phenomenological studies of QCD at intermediated energies

where the theory is strongly coupled. For reviews see [23–26]. Yet to my knowledge there is no clear derivation of the model starting from QCD (see [27,28] for early discussions of the NJL model as an effective description of QCD). Since the theory is nonrenormalizable and the cutoff scale is close to the scale determining the four-fermion coupling, it is not clear whether the model actually leads to any kind of controlled approximation. On the other hand, the non-local version discussed here is embedded in string theory which provides it with a UV completion. An improved understanding of the embedding of the NJL model in string theory may lead to a better understanding of its range of validity.

6.2. The Scales of Chiral Symmetry Breaking and Confinement.

To connect the $D4 - D8 - \overline{D8}$ model to four-dimensional physics we compactified the fourth directions, $x^4 \simeq x^4 + 2\pi R_4$. The model is then characterized by two dimensionless parameters, $\lambda_4 = \lambda/2\pi R_4$ which is the four-dimensional gauge coupling, and λ/L with L the separation between the $D8$ and $\overline{D8}$ -branes. This one more dimensionless parameter than is thought to exist in QCD. By dimensional transmutation we can turn λ_4 in to Λ_{QCD} which determines the scale of confinement. On the other hand, the strong coupling expression for U_0 shows that λ/L determines the scale of chiral symmetry breaking. Thus these two scales can be varied independently, subject to the obvious constraint that $L \leq \pi R_4$.

In real QCD it appears that the scales of chiral symmetry breaking and confinement are quite close to each other. Is this necessarily the case, or it possible that there is some deformation of QCD which would allow one to vary the scales separately? One way that one might imagine addressing this question is through lattice QCD with the addition of explicit four-fermion interactions which should mock up the higher-dimensional gauge boson exchange which in this model drives the chiral symmetry breaking.

6.3. Holography for Non-Conformal Theories and $M5$ -branes.

It has been known since the work of [13] that there exists a gauge/gravity correspondence for certain nonconformal theories which generalizes the well-known AdS/CFT correspondence related to $D3$ -branes. In particular, there exists such a correspondence for $D4$ -branes which is the basis for the dual description which was used here to study chiral symmetry breaking at strong coupling. However, what was used here, involving the background geometry of $D4$ -branes, is only part of the story. In the far UV (that is at very large U) the $D4$ -description breaks down and must be replaced by the geometry of wrapped $M5$ -branes. To my knowledge, the detailed mapping of fields and operators for such theories has not been worked out to the same level of detail as in AdS/CFT.

6.4. Role of the Tachyon

In both of the systems described here the gauge fields on the flavor D-branes are dual to the left-handed and right-handed quark currents, $\bar{q}_L \gamma^\mu q_L$, $\bar{q}_R \gamma^\mu q_R$. On the other hand, chiral symmetry breaking is determined by the vacuum expectation value of the operator $\bar{q}^\alpha q^\beta$. The bulk field which is dual to this operator is clearly the “tachyon” which stretches between the Dp and $\overline{Dp}$ -branes. Of course for large enough L this field is not actually tachyonic, it has a positive mass squared in the UV. The fact that chiral symmetry is broken is something that has been described indirectly here through the joining of the branes, but it should have a more direct manifestation in terms of the coefficient of the normalizable mode of this tachyon field at large U . Unfortunately it is now yet known how to include this tachyon field in a controlled approximation, although see [29] for some recent progress in this direction.

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REFERENCES

1. J. Polchinski, “Dirichlet-Branes and Ramond-Ramond Charges,” *Phys. Rev. Lett.* **75**, 4724 (1995) [arXiv:hep-th/9510017].
2. O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” *Phys. Rept.* **323**, 183 (2000) [arXiv:hep-th/9905111].
3. G. ’t Hooft, “Symmetry breaking through Bell-Jackiw anomalies,” *Phys. Rev. Lett.* **37**, 8 (1976).
4. Y. Nambu and G. Jona-Lasinio, “Dynamical model of elementary particles based on an analogy with superconductivity. II,” *Phys. Rev.* **124**, 246 (1961).
5. D. J. Gross and A. Neveu, “Dynamical Symmetry Breaking In Asymptotically Free Field Theories,” *Phys. Rev. D* **10**, 3235 (1974).
6. B. Berg and P. Weisz, “Exact S Matrix Of The Chiral Invariant SU(N) Thirring Model,” *Nucl. Phys. B* **146**, 205 (1978).
7. N. Andrei and J. H. Lowenstein, “Derivation Of The Chiral Gross-Neveu Spectrum For Arbitrary SU(N) Symmetry,” *Phys. Lett. B* **90**, 106 (1980).
8. N. Andrei and J. H. Lowenstein, “Diagonalization Of The Chiral Invariant Gross-Neveu Hamiltonian,” *Phys. Rev. Lett.* **43**, 1698 (1979).
9. R. Koberle, V. Kurak and J. A. Swieca, “Scattering Theory And 1/N Expansion In The Chiral Gross-Neveu Model,” *Phys. Rev. D* **20**, 897 (1979) [Erratum-ibid. *D* **20**, 2638 (1979)].
10. E. Witten, “Chiral Symmetry, The 1/N Expansion, And The SU(N) Thirring Model,” *Nucl. Phys. B* **145**, 110 (1978).
11. I. Affleck, “On The Realization Of Chiral Symmetry In (1+1)-Dimensions,” *Nucl. Phys. B* **265**, 448 (1986).
12. J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” *Cambridge, UK: Univ. Pr.* (1998) 531 p
13. N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” *Phys. Rev. D* **58**, 046004 (1998) [arXiv:hep-th/9802042].
14. A. Karch and E. Katz, “Adding flavor to AdS/CFT,” *JHEP* **0206**, 043 (2002) [arXiv:hep-th/0205236].
15. M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, “Towards a holographic dual of large-N(c) QCD,” *JHEP* **0405**, 041 (2004) [arXiv:hep-th/0311270].
16. J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik and I. Kirsch, “Chiral symmetry breaking and pions in non-supersymmetric gauge / gravity duals,” *Phys. Rev. D* **69**, 066007 (2004) [arXiv:hep-th/0306018].
17. T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” *Prog. Theor. Phys.* **113**, 843 (2005) [arXiv:hep-th/0412141].
18. T. Sakai and S. Sugimoto, “More on a holographic dual of QCD,” *Prog. Theor. Phys.* **114**, 1083 (2006) [arXiv:hep-th/0507073].
19. E. Antonyan, J. A. Harvey, S. Jensen and D. Kutasov, “NJL and QCD from string theory,” arXiv:hep-th/0604017.
20. E. Antonyan, J. A. Harvey and D. Kutasov, “The Gross-Neveu model from string theory,” arXiv:hep-th/0608149.
21. E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* **2**, 505 (1998) [arXiv:hep-th/9803131].
22. O. Aharony, J. Sonnenschein and S. Yankielowicz, arXiv:hep-th/0604161.
23. S. P. Klevansky, “The Nambu-Jona-Lasinio model of quantum chromodynamics,” *Rev. Mod. Phys.* **64**, 649 (1992).
24. T. Hatsuda and T. Kunihiro, “QCD phenomenology based on a chiral effective Lagrangian,” *Phys. Rept.* **247**, 221 (1994) [arXiv:hep-ph/9401310].
25. M. Buballa, “NJL model analysis of quark matter at large density,” *Phys. Rept.* **407**, 205 (2005) [arXiv:hep-ph/0402234].
26. M. K. Volkov and A. E. Radzhabov, “Forty-fifth anniversary of the Nambu-Jona-Lasinio

- model,” arXiv:hep-ph/0508263.
27. A. Dhar and S. R. Wadia, “The Nambu-Jona-Lasinio Model: An Effective Lagrangian For Quantum Chromodynamics At Intermediate Length Scales,” *Phys. Rev. Lett.* **52**, 959 (1984).
 28. A. Dhar, R. Shankar and S. R. Wadia, “Nambu-Jona-Lasinio Type Effective Lagrangian. 2. Anomalies And Nonlinear Lagrangian Of Low-Energy, Large N QCD,” *Phys. Rev. D* **31**, 3256 (1985).
 29. R. Casero, E. Kiritsis and A. Paredes, “Chiral symmetry breaking as open string tachyon condensation,” arXiv:hep-th/0702155.