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SUPERSTRINGS AND SOLITONS

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A low-energy analysis of macroscopic superstrings is presented and various analogies between these superstrings and solitons in supersymmetric theories are discussed. These include the existence of exact multi-string solutions of the low-energy supergravity super-Yang–Mills equations of motion and a Bogomol'nyi bound for the energy per unit length which is saturated by these solutions. Arguments are presented that these features will persist to all orders in α'. The connection between chiral fermion zero modes on the string and partially broken supersymmetry is also discussed.

1. Introduction

Most studies of classical solutions to superstring theory have focused on solutions which correspond to compactification of some number of spatial dimensions with the remaining dimensions being described by a flat Minkowski space-time. The correspondence between these solutions and superconformal field theory is now well understood. Recently there has been interest in expanding the study of classical solutions to include those with nontrivial dependence on the noncompact Minkowski coordinates as well [1–3].

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Unlike point theories or theories with open strings, purely closed-string theories such as the heterotic strings or type-II strings contain stable extended states which we will refer to as macroscopic strings [4]. These should correspond to classical solutions of string theory. Since they correspond to fundamental strings, one might argue that they are the most basic classical solutions to string theory. Although it would be of interest to construct these solutions directly as superconformal field theories, there is also much that may be learned by studying them at distances large compared to the string scale using the low-energy field-theory limit of string theory. That is what we will do in this paper.

In later sections we will study these macroscopic string states as singular solutions of the low-energy supergravity super-Yang–Mills lagrangian. The proper point of view regarding the singular nature of these solutions is not completely clear to us. On the one hand extended strings exist as fundamental quantum states which are visible in string perturbation theory. Therefore one should not expect to treat them as smooth solutions to the classical field equations any more than the electron is treated as a classical solution to the Maxwell equations. On the other hand we will find many similarities between macroscopic strings and solitons. Also, we will work in the limit where the massive modes of the string have been integrated out. Including the massive modes would certainly alter the short-distance structure of the solutions and might be expected to smooth out the singularity. A resolution of this issue will probably require a study of these solutions to all orders in α'.

From the low-energy point of view macroscopic strings have much in common with certain solitons which arise in supersymmetric field theories. For example, magnetic monopoles in the Bogomol'nyi–Prasad–Sommerfeld limit can be embedded into $N=2$ supersymmetric gauge theory. Another example is provided by the Papapetrou–Majumdar black holes in $N=2$ supergravity [5]. Many of the marvelous properties of these solutions such as the existence of exact multi-monopole solutions and the saturation of the Bogomol'nyi bound for the mass are then related to the underlying supersymmetry. We will find similar phenomena for macroscopic superstrings. An essential difference is that unlike ordinary solitons, which do not appear in perturbation theory, the ratio of the mass of our solutions to the mass of excited string states is independent of the string coupling constant. In addition, the "charge" associated with our string solutions is not topological but is more akin to electric charge in electromagnetism.

We will now give a brief outline of the rest of the paper. One of the themes we wish to develop is the correspondence between the field-theoretic description of macroscopic string states and string states as usually studied in first-quantized string theory. One result which led us to consider such a correspondence was the vanishing of the mass shift for certain types of string states. In sect. 2 we will describe the first-quantized string states we will consider and describe the calculation of their mass shift.

In sect. 3 we begin our discussion of macroscopic string solutions. We will present the field equations we wish to solve and discuss the derivation of the source terms in these equations. The ansatz we will use to solve these equations is related to a no-force condition which is analogous to that found by Manton for BPS monopoles [6], and is also related to the existence of unbroken supersymmetries. The supersymmetric structure of these solutions and the connection between supersymmetries and fermion zero modes on the string will be the subject of sect. 4. In sect. 5 we will show how the supersymmetry algebra can be used to demonstrate a Bogomol'nyi bound for this class of solutions and verify that our solutions saturate this bound. Sect. 6 is devoted to a brief discussion of our results.

2. Nonrenormalization of the superstring tension

We wish to study macroscopic strings in D-dimensional Minkowski space-time. The mass per unit length of such a state has a classical interpretation of string tension; thus from the mass shift we can extract the renormalization of the string tension. We assume that the additional $10-D$ dimensions required for a consistent superstring theory are replaced by a superconformal theory which results in at least $N=1$ space-time supersymmetry in D dimensions. Although the ideas in this paper could be generalized to study the excitation of fields associated with this compactification, here we will restrict ourselves purely to fields associated with the remaining Minkowski dimensions.

In order to study macroscopic strings in a well-defined way we will further compactify one of the remaining $D-1$ spatial dimensions on a circle of radius R. We will often consider R to be much larger than the string scale and regard strings with nonzero winding number on this circle to be macroscopic strings. Let k_{int} and $\bar{k}_{\text{int}}$ denote the right and left-moving momenta of the string on the circle respectively. In terms of the winding number L and the quantized momentum M/R with M integer we have $k_{\text{int}} = (M/2R + LR)$ and $\bar{k}_{\text{int}} = (M/2R - LR)$.

In string theory there will be an infinite tower of states of varying winding number L. Many of these will be unstable and will decay to lighter states consistent with conservation of L. We will be interested only in stable macroscopic states. It turns out that this still leaves an infinite tower of states which we now describe. For concreteness, consider the heterotic string. Then the right-moving ground state consists of a space-time vector and a space-time spinor coming from the Neveu–Schwarz and Ramond sectors respectively. The states we will consider are constructed as tensor products of the right-moving ground states with various left-moving states. If we let k_{M} denote the remaining Minkowski momentum and $\bar{N}$ the left-moving number operator, then these states obey a mass-shell condition

$$\tfrac{1}{2}\left(k_{\text{M}}^2 + k_{\text{int}}^2\right) = \tfrac{1}{2}\left(k_{\text{M}}^2 + \bar{k}_{\text{int}}^2\right) + \bar{N} - 1 = 0. \tag{2.1}$$

It follows from eq. (2.1) that for unit winding number ($L=1$) we have two solutions: ($M=-1, L=1, \bar{N}=0$) and ($M=0, L=1, \bar{N}=1$) of energy ($\frac{1}{4}R^2 + 1/R^2 - 1$) and $\frac{1}{4}R^2$ respectively. Thus, for a large enough radius, the former has the lowest energy. We can construct an infinite tower of stable bosonic states by adding various left-moving excitations to the right-moving ground state. For a given value of $\bar{N}$ these states have momenta and winding satisfying $\bar{N} = 1 - ML$ and energy $E = k_{\text{int}}^2 = (M/R + \frac{1}{2}LR)^2$. The right-moving part of the vertex operator for all these states is given by

$$W = \zeta_\mu \cdot (\partial X^\mu + ik \cdot \psi\psi^\mu) e^{ik \cdot X},$$

$$k = (k_{\mathrm{M}}, k_{\text{int}}), \qquad \zeta \cdot k = 0. \tag{2.2}$$

Note that this is the same as the right-moving part of the vertex operator for a massless gauge boson in $(D-1)+1$ dimensions.

The mass shifts and decay rates of massive states are related to the real and the imaginary part of the two-point amplitude. For a stable winding state with vertex operator V the mass shift is directly given by the two-point function. The two-point function A at order g is given by integrating the correlation function of vertex operators $\langle VV \rangle$ over the genus-g moduli space with the appropriate measure and summing over spin structures. The right-moving part of the vertex for the lightest winding state in the 0-picture, evaluated in its rest frame, is given by eq. (2.2). Without the correlator $\langle VV \rangle$, the one-loop amplitude A is just the cosmological constant at one-loop level and vanishes by supersymmetry after summing over spin structures. Thus, the nonzero part of the answer can come only from the spin structure dependent part of the correlation function. We therefore need to concentrate on the correlation function

$$\left\langle ik_1 \cdot \psi \zeta_1 \cdot \psi \, e^{ik_1 \cdot X} e^{i\bar{k}_1 \cdot \bar{X}}(\nu_1, \bar{\nu}_1) ik_2 \cdot \psi \zeta_2 \cdot \psi \, e^{ik_2 \cdot X} e^{i\bar{k}_2 \cdot \bar{X}}(\nu_2, \bar{\nu}_2) \right\rangle. \tag{2.3}$$

After summing over spin structures the one-loop amplitude is proportional to the kinematic factor

$$(\zeta_1 \cdot k_2 \zeta_2 \cdot k_1 - \zeta_1 \cdot \zeta_2 k_1 \cdot k_2), \tag{2.4}$$

which results from performing the fermion contractions. Since $k_1 = -k_2$ by momentum conservation, $\zeta_1 \cdot k_2 = \zeta_2 \cdot k_1 = k_1 \cdot k_2 = 0$, and hence this factor vanishes. As a result the mass shift vanishes at one-loop level.

At one-loop level, as discussed in refs. [7, 8], there are some theories in which this naïve argument fails as a result of infrared subtleties. In theories with anomalous U(1) factors which lead to supersymmetry breaking D terms there can be singularities arising from the integration over the relative position of the vertex operators which are essential to obtaining nonvanishing scalar masses. We have checked by explicit evaluation of the one-loop amplitude that such singularities do not arise here. The details of this calculation are given in appendix A. At higher-loop level there are many subtleties in superstring perturbation theory, and it is not clear that even the cosmological constant vanishes order by order in perturbation theory without adjustment of the string tension [9]. However we would expect that any formulation of superstring perturbation theory in which mass shifts for massless gauge bosons vanish (as would be expected by gauge invariance) should also lead to vanishing mass shifts for the tower of states discussed here since the right-moving part of the vertex operator for massless gauge bosons is the same as for these states. Note also that the vanishing of the imaginary part of the two-point function for this tower of states implies that they are all stable, since otherwise the two-point function would have to be an imaginary part by unitarity. The stability of these states at tree level can be easily checked.

The vanishing of the mass shift for the winding states suggests that the superstring tension does not get renormalized in perturbation theory. To make this statement more precise we would have to give a definition of the string tension in terms of dimensionless ratios of coupling constants and show that this ratio of coupling constants is not renormalized in perturbation theory. Alternatively, one might define the string tension in terms of the slope of the Regge trajectory for very massive states, i.e. in the classical limit. The mass shifts for the massive states in the leading Regge trajectory have been calculated [10] in more than four dimensions. For very massive states the renormalization of the slope does vanish in agreement with our result.

The nonrenormalization of the mass of these states is reminiscent of nonrenormalization theorems for solitons in supersymmetric theories. These nonrenormalization theorems are usually understood to be a consequence of the existence of central extensions of the supersymmetry algebra. Since our solutions occur in a supersymmetric theory, one might suspect a similar situation here. We will discuss this issue further in sect. 5. We first turn to a discussion of these macroscopic string states from the point of view of low-energy field theory.

3. Exact multi-string solutions

In this section we describe the exact stringlike solutions to the low-energy effective action for the massless modes of the string.

To find these solutions, we start from a sigma-model action S_σ that describes the coupling of a string to the metric $g_{\mu\nu}$, antisymmetric tensor or Kalb–Ramond field $B_{\mu\nu}$ and dilaton ϕ. One usually thinks of this action as describing a string moving in a background provided by the massless modes of the string. Demanding conformal invariance of the sigma model then leads to the equations of motion for these background fields. Our point of view will be somewhat different. We wish to obtain the equations satisfied by the massless fields in the presence of a string source. We will thus use S_σ to evaluate the *source terms* for the background fields in the presence of a macroscopic string source. This could also be done by simple vertex operator calculations as in ref. [4].

Following the treatment in ref. [11] it is convenient to perform a conformal rescaling $g_{\mu\nu} \to g_{\mu\nu}\, e^{4\phi/(D-2)}$ in D space-time dimensions. We can then incorporate the equations of motion which follow from conformal invariance and the classical couplings of the bosonic massless fields to the string in the combined action

$$S = \frac{1}{2\kappa^2}\int d^D x \sqrt{g}\,\mathscr{L}_B + S_\sigma\,, \tag{3.1}$$

with the bosonic lagrangian given by

$$\mathscr{L}_B = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{-2\alpha\phi}H^2\,, \tag{3.2}$$

and the sigma-model action by

$$S_\sigma = -\tfrac{1}{2}\mu\int d^2\sigma\left(\sqrt{\gamma}\,\gamma^{mn}\,\partial_m X^\mu\,\partial_n X^\nu\,g_{\mu\nu}\,e^{\alpha\phi} + \epsilon^{mn}\,\partial_m X^\mu\,\partial_n X^\nu\,B_{\mu\nu}\right). \tag{3.3}$$

Here we have rescaled the dilaton such that it has the standard kinetic term, and $\alpha \equiv \sqrt{2/(D-2)}$. The equations of motion that follow from the above action are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}\,, \tag{3.4a}$$

$$\nabla_\rho(e^{-2\alpha\phi}H^{\mu\nu\rho}) = 2\mu\kappa^2\int d^2\sigma\,\epsilon^{mn}\,\partial_m X^\mu\,\partial_n X^\nu\,\frac{\delta^{(D)}(x-X)}{\sqrt{g}}\,, \tag{3.4b}$$

$$\nabla_\mu(g^{\mu\nu}\,\nabla_\nu\phi) = -\tfrac{1}{6}\alpha\,e^{-2\alpha\phi}H^2 + \mu\kappa^2\alpha\int d^2\sigma\sqrt{\gamma}\,\gamma^{mn}\,\partial_m X^\mu\,\partial_n X^\nu\,g_{\mu\nu}\,e^{\alpha\phi}\,\frac{\delta^{(D)}(x-X)}{\sqrt{g}}\,. \tag{3.4c}$$

Here $T_{\mu\nu}$ includes the usual energy–momentum tensor for the antisymmetric tensor and dilaton fields as well as the energy–momentum tensor corresponding to the string source:

$$2\kappa^2 T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}(\partial\phi)^2 + \tfrac{1}{2}\left(H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - \tfrac{1}{6}g_{\mu\nu}H^2\right)e^{-2\alpha\phi} - 2\mu\kappa^2\int d^2\sigma\sqrt{\gamma}\,\gamma^{mn}\,\partial_m X_\mu\,\partial_n X_\nu\,e^{\alpha\phi}\,\frac{\delta^{(D)}(x-X)}{\sqrt{g}}\,. \tag{3.5}$$

This is a complicated, nonlinear system of equations, and it would seem miraculous to be able to find an exact solution. There is however a remarkably simple ansatz that reduces these equations to one *linear* differential equation for a *single* scalar function $E(r)$:

$$\mathrm{d}s^2 = \mathrm{e}^{A}\left[-\mathrm{d}t^2 + (\mathrm{d}x^1)^2\right] + \mathrm{e}^{B}\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x},$$

$$A = \frac{D-4}{D-2}E(r), \qquad B = -\frac{2}{D-2}E(r), \qquad \phi = \alpha E(r), \qquad B_{01} = -\mathrm{e}^{E(r)}. \tag{3.6}$$

Here the string runs along the x^1 axis and $r^2 = \boldsymbol{x}\cdot\boldsymbol{x} = \eta_{ij}x^i x^j$, $i, j = 2, \dots, D-1$. Note that with this ansatz the transverse sigma-model metric $G_{ij} \equiv g_{ij}\,\mathrm{e}^{\alpha\phi}$ is flat.

This ansatz can be motivated in different ways. As we will see in sect. 4, these relations between the different fields ensure that half of the supersymmetries will be preserved. Related relations were considered in the work of Strominger [12]. Another more physical point of view is related to the vanishing of the force between parallel strings. This is an important feature of our solution that is true in all dimensions and is necessary in order that there exist exact, stable multi-string solutions. An analogous situation arises in the case of BPS monopoles [6]. The classical force between two widely separated monopole configurations has two contributions. There is an attractive Coulomb force between the two monopoles due to scalar exchange. There is also a repulsive (attractive) Coulomb force between them due to vector exchange if the monopoles have equal (opposite) magnetic charge. In the BPS limit these two contributions have the same magnitude. Therefore there is no net classical force between two monopoles that have equal charge and we can view the Bogomol'nyi equation as a "no-force condition" that the net force on a test monopole in the background of the field produced by a source monopole vanishes.

There is a perfect analog of Manton's condition for our solutions. The equations of motion that follow from eq. (3.1) are

$$\nabla_m(\gamma^{mn}\nabla_n X^\mu) = -\Gamma^\mu_{\nu\rho}\partial_m X^\nu \partial_n X^\rho \gamma^{mn} + \tfrac{1}{2}H^\mu{}_{\nu\rho}\partial_m X^\nu \partial_n X^\rho \epsilon^{mn}. \tag{3.7}$$

Here $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols calculated using the rescaled sigma-model metric $G_{\mu\nu} = g_{\mu\nu}\,\mathrm{e}^{\alpha\phi}$; this term therefore incorporates the contribution to the force from both the graviton and the dilaton.

Now, consider a stationary test string at some distance from a source string located at the origin. Both strings run along the x^1 direction and have the same orientation. The field produced by the source string at the test string is given by eq. (3.6). Using conformal gauge for the test string we have $X^0 = \tau$, $X^1 = \sigma$, $\gamma_{mn} =$

diag$(-1,+1)$, and $\epsilon^{01}=+1$. In order that the net transverse force on the test string vanish we require

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}X^i = 2\Gamma^i_{01} + H^i{}_{01} = 0. \tag{3.8}$$

This gives $H_{i01} \equiv \partial_i B_{01} = \partial_i G_{00} = -\partial_j(\mathrm{e}^A\,\mathrm{e}^{\alpha\phi})$, which is indeed true for our ansatz because $A+\alpha\phi = E$ and $B_{01} = -\mathrm{e}^E$. On the other hand, if the test string has orientation opposite to that of the source string, then $X^1 = -\sigma$ and the term H^i_{01} in eq. (3.8) appears with a negative sign. Therefore, there is a net attractive force and the two strings would annihilate each other. The no-force condition is not sufficient to determine the functional form of $E(r)$ at the level that we are working. To do this we must investigate the full equations of motion.

To solve the equations of motion we first use eq. (3.6) to write the energy-momentum tensor in the following form:

$$T_{\alpha\beta}(\phi,H) = -(1/4\kappa^2)(\alpha^2+1)\mathrm{e}^A\eta^{ij}\partial_i E\,\partial_j E\,g_{\alpha\beta},$$

$$T_{ij}(\phi,H) = (1/2\kappa^2)(\alpha^2-1)\left[\partial_i E\,\partial_j E - \tfrac{1}{2}\eta_{ij}(\partial E)^2\right],$$

$$T_{\alpha\beta}(\text{source}) = -\left(\mu/\sqrt{g}\right)\int \mathrm{d}^2\sigma\,\sqrt{\gamma}\,\gamma^{mn}\,\partial_m X_\alpha\,\partial_n X_\beta\,\mathrm{e}^{\alpha\phi}\,\delta^{(D)}(x-X). \tag{3.9}$$

Here the indices α,β are in the (x^0,x^1) plane, i,j are in the directions transverse to the string, and η_{ij} is the flat-space metric in the transverse dimensions. Now define $S_{\mu\nu} = T_{\mu\nu} - [1/(D-2)]g_{\mu\nu}T$, where T is the trace of the energy–momentum tensor. Then with our ansatz the Einstein equations are automatically satisfied as an identify for any choice of the function $E(r)$,

$$R_{ij} \equiv S_{ij} = -\frac{D-4}{2(D-2)}\partial_i E\,\partial_j E + \frac{1}{D-2}\eta_{ij}(\partial E)^2 + \frac{1}{D-2}2\mu\kappa^2\,\mathrm{e}^E\eta_{ij}\,\delta^{(D-2)}(x), \tag{3.10}$$

$$R_{\alpha\beta} \equiv S_{\alpha\beta} = \mathrm{e}^{-A}g_{\alpha\beta}\left[-\frac{D-4}{2(D-2)}(\partial E)^2 - \frac{D-4}{(D-2)}\mu\kappa^2\,\mathrm{e}^{DE/(D-2)}\,\delta^{(D-2)}(x)\right]. \tag{3.11}$$

The equations of motion for the dilaton (3.4b) and the antisymmetric tensor field (3.4c) both reduce to the same linear equation:

$$\eta^{ij}\partial_i\partial_j\,\mathrm{e}^{-E} = -2\mu\kappa^2\,\delta^{(D-2)}(x), \tag{3.12}$$

with solution

$$e^{-E} = \begin{cases} 1 + \dfrac{M}{r^{D-4}}, & D > 4, \quad (3.13a) \\ 1 - 8G\mu \ln(r), & D = 4, \quad (3.13b) \end{cases}$$

where $M = 2\kappa^2\mu/(D-4)\omega_{D-3}$ and ω_{D-3} is the volume of the sphere S^{D-3}.

Since the solution satisfies the no-force condition, it is straight forward to write down an exact, stable multi-string solution that represents strings located at various positions $\{\boldsymbol{r}_l\}$. From eqs. (3.13a, b) we see that such a solution is given by a linear superposition

$$e^{-E} = \begin{cases} 1 + \sum\limits_l \dfrac{M}{|\boldsymbol{r} - \boldsymbol{r}_l|^{D-4}}, & D > 4, \quad (3.14a) \\ 1 - \sum\limits_l 8G\mu \ln(|\boldsymbol{r} - \boldsymbol{r}_l|), & D = 4. \quad (3.14b) \end{cases}$$

Let us now discuss in some detail the solution in $D = 4$, partly because it has some qualitatively different features and also because it is easier to visualize and will give us a better understanding of these exact solutions. Let us first recall a simpler exact solution in four dimensions, viz. the gravitational field around a cosmic gauge string. It is well known that space around a gauge string is conical with deficit angle $8\pi G\mu$. The energy–momentum tensor for such a cosmic string source has the special form $T_{\mu\nu} = \mathrm{diag}(\rho, -\rho, 0, 0)$. The crucial point is that the energy–momentum tensor is zero in the transverse plane and is proportional to the Minkowski metric in the (x^0, x^1) plane. As a result we can take the four-dimensional space-time to be of the form $M^4 = M^2 \times \Sigma^2$ where M^2 is a flat Minkowski (x^0, x^1) plane and Σ^2 is a curved transverse two-dimensional surface. Thus a four-dimensional problem is effectively reduced to a two-dimensional problem. But in two dimensions, the Einstein equation is an identity since the Einstein–Hilbert action is purely topological, i.e. invariant under metric variations; thus $R_{ij} \equiv \frac{1}{2}g_{ij}R$. The equations in the (x^0, x^1) plane reduce to $-\frac{1}{2}R\eta_{\alpha\beta} = -8\pi G\rho\eta_{\alpha\beta}$, giving us

$$R = 16\pi G\rho. \tag{3.15}$$

One can regard this equation as a linear second-order differential equation for the conformal factor of the two-dimensional metric, which can be easily solved. In fact, for a cosmic gauge string source the energy density is nonzero only at the core of the string, hence by eq. (3.15) the Ricci scalar is nonzero only at the origin. In two dimensions the Riemann tensor is proportional to the Ricci scalar; we thus have a space that is flat everywhere except for a curvature singularity at the origin, i.e. a conical space.

Now, let us look at the field around a cosmic superstring. In four dimensions, the antisymmetric tensor field is equivalent to an axion field by a duality transformation. A cosmic superstring is therefore analogous to a global axion string. For a global string, unlike for a gauge string, there is nonzero energy density and pressure outside the string core because there is a non-zero axion field outside the core. As a result we cannot take the space to be of the form $M^4 = M^2 \times \Sigma^2$ and we have to solve the full four-dimensional Einstein equations. The gravitational field around a global axion string has been calculated in the linearized approximation [13] and the exact solution was found in ref. [14]. A cosmic superstring is a bit different in that it couples not only to the axion and graviton but also to the dilaton. This apparent complication actually simplifies the solution dramatically. The dilaton field around a string depends only on r, the radial distance from the core of the string. As a result, the energy–momentum tensor due to the dilaton has pressure in the r direction and tension in the θ direction. The axion field, on the other hand, regarded as an angular variable, is proportional to the angle θ around the string. Its energy–momentum tensor, therefore, has pressure in the θ direction and tension in the r direction. In string theory the coefficients are such that the tension (= negative pressure) due to the dilaton precisely cancels the pressure due to the axion and vice versa. Since both contribute the same energy density and negative pressure in the x^1 direction, the net energy-momentum tensor even outside the core of a cosmic superstring has the same simple form as before $T_{\mu\nu} = \mathrm{diag}(\rho, -\rho, 0, 0)$ (see eq. (3.9) for $D = 4$). We can now repeat the analysis described in the last paragraph for our cosmic superstring. We will again get a product space $M^4 = M^2 \times \Sigma^2$ with the Ricci scalar for Σ^2 given by eq. (3.15). The only difference is that now we have a nonzero energy density outside the string; as a result Σ^2 is not a simple conical space. It is straightforward, however, to solve the equation for the conformal factor of the metric for Σ^2 and we get the solution described before.

What happens in higher dimensions? When we have more than two transverse dimensions, there is no geometric reason why the Einstein equations should reduce to anything simple. But with our ansatz, the Einstein equations do once again reduce to an identity and we are left with only a linear equation (3.12) that follows from the equation of motion of the axion and the dilaton.

Another interesting property of this solution is the cancellation of self-energies in four dimensions. In sect. 2 we saw that the quantum correction to the self-energy is zero. On the other hand, classically we know that a global string like a cosmic superstring ought to have logarithmically divergent self-energy because of the axion field around it. This infrared divergence should show up at the quantum level to give a divergent answer. But since the quantum answer is zero, the classical answer must also be zero. This is indeed true. There are three separate divergent contributions to the classical self-energy. The dilaton contribution is the same as the axion but the graviton contribution is negative and twice as large. The net classical self-energy thus adds up to zero in agreement with the quantum answer [15].

We now discuss this point in more detail. In general relativity, there is no unique way to define the local energy density of the gravitational field even in the weak-field limit. In what follows, we will adopt the following, standard definition of the gravitational energy–momentum pseudo-tensor. Let us write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In $D > 4, h_{\mu\nu}$ is asymptotically zero but not necessarily small everywhere. This allows a satisfactory definition of "total" ADM mass per unit length for our asymptotically flat solution. In four dimensions $h_{\mu\nu}$ is asymptotically logarithmically divergent. However, we still adopt this definition because this is what we will see in a perturbative string calculation which is performed around a flat Minkowski background space. With this separation one can regard the part of the Einstein tensor that is linear in $h_{\mu\nu}$ as the kinetic term and the remaining as the gravitational energy–momentum pseudo-tensor. One can then rewrite the Einstein equation as

$$R^{(1)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} R^{(1)\rho}_{\rho} = \kappa^2 \Theta_{\mu\nu} . \tag{3.16}$$

One can take this as the *definition* of the "total" energy–momentum tensor Θ. The linear Ricci tensor is given by

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 h^\rho_\mu}{\partial x^\rho \partial x^\nu} + \frac{\partial^2 h^\rho_\nu}{\partial x^\rho \partial x^\nu} - \frac{\partial^2 h^\rho_\rho}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\rho \partial x_\rho} \right), \tag{3.17}$$

where the indices are raised and lowered using the flat Minkowski metric. Using this expression for $R^{(1)}_{\mu\nu}$, it is straightforward to calculate the total energy density for our solution

$$\Theta_{\alpha\beta} = \frac{1}{2\kappa^2}\eta_{\alpha\beta}\left[\frac{\partial^2 \mathrm{e}^A}{\partial x^2} + (D-3)\frac{\partial^2 \mathrm{e}^B}{\partial x^2} \right]. \tag{3.18}$$

In four dimensions $A = 0$ and $B = -E$, we thus find

$$\Theta_{00} = \mu\, \delta^{(2)}(x) . \tag{3.19}$$

We see that the net energy density outside the string core vanishes as a result of a precise cancellation between the graviton, dilaton and axion as expected from the quantum calculation in the last section.

In higher dimensions, the metric is asymptotically flat; as a result we can express the total energy of the solution in terms of a surface integral that depends only on the asymptotic properties of the metric. From eq. (3.16) we have

$$\text{mass/length} = \int \mathrm{d}^{D-2}x\, \Theta_{00} = \frac{1}{\kappa^2}\int \mathrm{d}^{D-2}x\left[R^{(1)}_{00} - \tfrac{1}{2}\eta_{00}R^{(1)\rho}_{\rho}\right]$$

$$= \frac{1}{2\kappa^2}\int\left(\frac{\partial h_{ij}}{\partial x^j} - \frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{11}}{\partial x^i}\right)n^i r^{D-3}\,\mathrm{d}\Omega_{D-3}\,. \tag{3.20}$$

This definition gives the ADM mass per unit length. This is the quantity that will enter into our later discussion of the generalized positive mass theorem and global supersymmetry algebra in the background of a macroscopic string. Substituting our solution in this expression gives

$$\text{mass/length} = -\frac{1}{2\kappa^2}\int_{\partial M}\left[\frac{\partial\,\mathrm{e}^{A}}{\partial x^i} + (D-3)\frac{\partial\,\mathrm{e}^{B}}{\partial x^i}\right]\mathrm{d}\Sigma^i\,. \tag{3.21}$$

Using the fact that $A+(D-3)B = -E$ and the asymptotic form of E we see that the ADM mass per unit length is identical to the string tension μ.

Before ending this section, we would like to point out some similarities and differences between our strings and the stringy cosmic strings discussed by Greene et al. [3, 16]. Unlike the stringy cosmic strings our solutions exist in arbitrary dimensions and describe macroscopic *fundamental* strings. In four dimensions we can express the antisymmetric tensor field in terms of a pseudoscalar by a dual transformation and then combine the axion, dilaton and the dilatino into a complex superfield. We can then cast our solution as a σ-model string [17] in $N=1$ supergravity with the Kähler potential $K(\Phi,\Phi^*) = -\ln(\Phi+\Phi^*)$. This Kähler potential defines a metric of constant negative curvature on the upper-half plane whose complex coordinate is $u = B - i\,\mathrm{e}^{-\phi}$, where B is the axion field related to $H_{\mu\nu\rho}$ by $\epsilon^{\mu\nu\rho\sigma}\partial_\mu B = \mathrm{e}^{-2\phi}H^{\nu\rho\sigma}$. To solve the sigma-model field equations it is sufficient to take u to be a holomorphic or anti-holomorphic function of x^1+ix^2. This condition ensures that the spatial components of the energy–momentum tensor vanish. To match the source the appropriate holomorphic solution for a single string is given by eq. (3.13b) and for a multi-string by eq. (3.14b). Were it not for the singularity at a finite distance we could derive a Bogomol'nyi bound in the manner described in ref. [17]. Moreover, the holomorphic solutions (3.13b) and (3.14b) are locally supersymmetric. The Kähler potential used by Greene et al. [3] is the same as above but with a crucial difference in the interpretation. They take as the dilaton component of the superfield the internal dilaton that arises from compactification, whereas ours is the space-time dilaton. As a result, they have modular invariance in the target space and the target space is the moduli space of the torus and not the upper-half plane. This changes the character of the solution completely. There solution "decompactifies" at infinity, whereas our solution has a nice asymptotic limit (in $D>4$). Also, we have an exact multi-string solutions for any number of strings whereas they have exact solutions for a maximum of twelve strings. The supersymmetric properties of our solution for $D>4$ will be discussed in the following two sections.

4. Killing spinors, space-time supersymmetry and fermion zero modes

The bosonic lagrangian of eq. (3.2) can be coupled to fermionic matter via supergravity theories. For example, one could think of it as the bosonic content of $N=1, D=10$ supergravity. Then, as we show below, the solution presented in sect. 3 admits Killing spinors, or in other words, constitutes a supersymmetric purely bosonic background. The lagrangian (3.2) can also be used to describe the (graviton, scalar field, pseudoscalar field) sector of $N=4, D=4$ supergravity, with the pseudoscalar field, B, given in terms of the antisymmetric tensor by $\epsilon^{\mu\nu\rho\sigma}\partial_\mu B = \mathrm{e}^{-2\phi} H^{\nu\rho\sigma}$. Again, the solution of sect. 3, with all the other fields in the theory set equal to zero, provides a supersymmetric solution. The proof in both cases is by inspection; here we present the details for the first case.

The lagrangian density of $N=1, D=10$ supergravity as given in ref. [18] can be written, after suitable redefinitions of the bosonic matter fields, in the following way:

$$\mathscr{L} = \mathscr{L}_\mathrm{B} + \mathscr{L}_\mathrm{F} + \mathscr{L}_\mathrm{I}\,, \tag{4.1}$$

where $\mathscr{L}_\mathrm{B}$ is given by eq. (3.2) with $D=10$, $\mathscr{L}_\mathrm{F}$ is the free lagrangian for the spin-$\frac{1}{2}$ field λ and the spin-$\frac{3}{2}$ field ψ_μ,

$$\mathscr{L}_\mathrm{F} = -\tfrac{1}{2}\left(\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \bar{\lambda} \not{D} \lambda\right), \tag{4.2}$$

and $\mathscr{L}_\mathrm{I}$ is the interaction piece, all whose terms are at least quadratic in Fermi fields. It is obvious that $g_{\mu\nu}, \phi, H_{\mu\nu\rho}$ as given by eq. (3.6) and $\psi_\mu = \lambda = 0$ is a solitonic solution to the classical field equations. We want to prove that this solution is supersymmetric. One way to do this is to verify that the variations of the fields under a supersymmetry transformation with parameter ϵ vanish.

Now, in a purely bosonic background the only fields whose variations are not zero are the dilatino λ and the gravitino ψ_μ, for which

$$\delta_\epsilon \lambda = -\frac{\sqrt{2}}{4\kappa}(\not{\partial}\phi)\epsilon + \frac{1}{24\sqrt{2}\,\kappa}\mathrm{e}^{-\phi/2} H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon\,, \tag{4.3a}$$

$$\delta_\epsilon \psi_\mu = (1/\kappa)\hat{D}_\mu \epsilon\,, \tag{4.3b}$$

with $\hat{D}_\mu$ the supercovariant derivative in the bosonic background acting on spin-$\frac{1}{2}$

fields,

$$\hat{D}_\mu \epsilon = \left(D_\mu + A_\mu\right)\epsilon , \tag{4.4}$$

$$A_\mu = \tfrac{1}{96} e^{-\phi/2} H_{\nu\rho\sigma}\left(\gamma_\mu{}^{\nu\rho\sigma} - 9\,\delta_\mu{}^\nu \gamma^{\rho\sigma}\right). \tag{4.5}$$

For the background to be supersymmetric the right-hand sides of eqs. (4.3a) and (4.3b) have to vanish. So for supersymmetry one has to solve the following two equations in ϵ:

$$\left[-(\partial\!\!\!/\phi) + \tfrac{1}{12} e^{-\phi/2} H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right]\epsilon = 0 , \qquad \hat{D}_\mu \epsilon = 0 , \tag{4.6a, b}$$

whose solutions are called Killing spinors.

It is not difficult to see that for the solution found in sect. 3 the equations for the Killing spinors reduce to

$$\left(\Gamma^j \partial_j e^{-\phi_s/2}\right)\left(1 + \Gamma_0 \Gamma_1\right)\epsilon = 0 . \tag{4.7}$$

$$\left[\partial_0 - \tfrac{3}{8}\left(\Gamma^j \partial_j e^{\phi_s}\right)\Gamma_0\left(1 + \Gamma_0\Gamma_1\right)\right]\epsilon = 0 , \tag{4.8a}$$

$$\left[\partial_1 + \tfrac{3}{8}\left(\Gamma^j \partial_j e^{\phi_s}\right)\Gamma_0\left(1 + \Gamma_0\Gamma_1\right)\right]\epsilon = 0 , \tag{4.8b}$$

$$\left\{\partial_i + \tfrac{1}{2}\left[(\partial_i\phi_s) - \tfrac{1}{4}\Gamma_i\left(\Gamma^j\partial_j\phi_s\right)\right]\left(1 + \Gamma_0\Gamma_1\right)\right\} e^{-3\phi_s/8}\epsilon = 0 , \tag{4.8c}$$

where ϕ_s is given by eqs. (3.6),(3.13a) with $D = 10$, and Γ_a denote the gamma matrices in flat space-time. The solution to eqs. (4.7, (4.8a–c) is

$$\epsilon = e^{3\phi_s/8}\epsilon_0 , \tag{4.9}$$

with ϵ_0 a constant spinor satisfying $(1 + \Gamma_0\Gamma_1)\epsilon_0 = 0$. This last condition is not satisfied by all the supersymmetries, which amounts to saying that the solitonic background has partially broken supersymmetry.

To discuss this partial breaking of supersymmetry and what it implies for the fermion zero modes we look at eq. (4.7), which came from vanishing of the variation of the dilatino. In $D = 10$, ϵ is Majorana–Weyl and therefore has definite chirality, $\Gamma_0\Gamma_1 \ldots \Gamma_9 = -1$ say. It then follows that the transverse or SO(8) chirality and the chirality along the string are correlated, that is $\Gamma_0\Gamma_1 = \Gamma_2\Gamma_3 \ldots \Gamma_9$. This, together with (4.8a–c), implies that the supersymmetries of the macroscopic string solution are those with negative chirality in the transverse directions. Those supersymmetries which do not annihilate the solution lead to fermion zero modes along the string. Thus the physical zero modes will be right-moving on the string and will transform as a spinor of SO(8). By the argument given in ref. [19], the low-energy action for the coupled system of the transverse string collective coordinates X^i and these fermion zero modes S^a_+ will be given by the Green–Schwarz action in "static" gauge, (i.e. $X^1 = \sigma, X^0 = \tau$). From this point of view, the string super-coordinates are just the world-sheet Goldstone bosons and Goldstone fermions associated with the broken translational invariance and global supersymmetry.

Here we started with $N = 1$ supergravity and we obtained a chiral Green–Schwarz superstring as the solution. If we had started with $N = 2$ supergravity we would have found twice as many fermion zero modes and would have obtained type-IIA or type-IIB superstrings depending on the chirality of the original supergravity theory*. We thus find a satisfying picture of how the Green–Schwarz fermions of the first quantized string theory are described in terms of low-energy fields.

5. Bogomol'nyi inequality

Partial breaking of supersymmetry is related to the existence of central charges in the global supersymmetry algebra. In what follows we prove that the supersymmetric solution of the $N = 1, D = 10$ supergravity equations discussed in the previous section has a central charge T responsible for the breaking of supersymmetry. The existence of such terms was first discussed in ref. [19] in the context of cosmic string solutions in globally supersymmetric theories; in the case of extended objects their relation to a Wess–Zumino term in the effective action was studied in ref. [20]. We will use the techniques of refs. [21] and [22] to derive from the centrally extended supersymmetry algebra a Bogomol'nyi inequality for a general class of solutions. We will further show that for our solution this inequality is saturated. Since, as pointed out in ref. [2], the solutions of ref. [21] may also be considered as solutions of the string-field equations the methods we are about to describe would seem to have a rather wider range of applicability than the case discussed below.

Let us consider $N = 1, D = 10$ supergravity. Since we are interested in discussing stringlike solutions, we will consider a general class of spaces that are translationally invariant in one direction, say along the x^1 direction. The supercharge per unit length $Q[\epsilon']$ associated to a supersymmetry transformation of commuting spinor parameter ϵ' is then

$$Q[\epsilon'] = \frac{4}{\kappa} \int_{\partial\Sigma} \bar{\epsilon}' \gamma^{\mu\nu\rho} \psi_\rho \, \mathrm{d}\Sigma_{\mu\nu} , \tag{5.1}$$

with Σ an eight-dimensional spacelike surface. If we now vary $Q[\epsilon']$ under a supersymmetry transformation whose parameter is another commuting spinor ϵ we

* We would like to thank A. Strominger for useful discussions on this point.

obtain

$$\delta_\epsilon Q[\epsilon'] = \frac{4}{\kappa^2}\int_{\partial\Sigma} N^{\mu\nu}\,\mathrm{d}\Sigma_{\mu\nu} = \frac{8}{\kappa^2}\int_\Sigma \nabla_\mu N^{\mu\nu}\,\mathrm{d}\Sigma_\nu\,, \tag{5.2}$$

where

$$N^{\mu\nu} = \bar{\epsilon}'\gamma^{\mu\nu\rho}\hat{D}_\rho\epsilon \tag{5.3}$$

is Nester's form [23] and $\hat{D}_\rho$ is the supercovariant derivative. Here we are interested in bosonic backgrounds, for which $\hat{D}_\rho$ is given by eqs. (4.4) and (4.5). After considerable algebra we find that the divergence of Nester's form can be written as

$$\begin{aligned}\nabla_\mu N^{\mu\nu} = {} & \overline{\hat{D}_\mu\epsilon'}\gamma^{\mu\nu\rho}\hat{D}_\rho\epsilon + \kappa^2\overline{(\delta_{\epsilon'}\lambda)}\,\gamma^\nu(\delta_\epsilon\lambda) \\ & - \tfrac{1}{2}\left[G^{\nu\rho} - \kappa^2 T^{\nu\rho}(\phi, H)\right]\bar{\epsilon}'\gamma_\rho\epsilon - \tfrac{1}{4}\mathrm{e}^{\phi/2}\left(\nabla_\sigma \mathrm{e}^{-\phi}H^{\sigma\nu\rho}\right)\bar{\epsilon}'\gamma_\rho\epsilon\,, \end{aligned} \tag{5.4}$$

where $G^{\nu\rho}$ is the Einstein tensor and $T^{\nu\rho}(\phi, H)$ is the energy–momentum tensor of the scalar field and the antisymmetric tensor field.

We next evaluate the integrals entering in eq. (5.2) for a general class of solutions; those with the following fall-off at infinity ($r \to \infty$):

$$\mathrm{d}s^2 \xrightarrow[r\to\infty]{} u^2\left[-\mathrm{d}t^2 + (\mathrm{d}x^1)^2\right] + v^2\,\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}\,, \tag{5.5a}$$

$$u^2 = 1 - \frac{3}{4}\frac{M}{r^6}\,, \qquad v^2 = 1 + \frac{1}{4}\frac{M}{r^6}\,, \tag{5.5b,c}$$

$$H = \frac{6Z}{r^6}\mathrm{d}r\wedge\mathrm{d}t\wedge\mathrm{d}x^1\,, \qquad \phi = -\frac{Y}{2r^6}\,, \tag{5.5d,e}$$

where M is proportional to the ADM mass per unit length and Z and Y are arbitrary. This behaviour is required by the finiteness of the action (3.1), or alternatively by the field equations at infinity. Assuming that $\epsilon' \to \epsilon_0'$ and $\epsilon \to \epsilon_0$ as $r \to \infty$, with ϵ_0 and ϵ_0' constant commuting spinors, we get on the one hand that

$$\begin{aligned}\frac{4}{\kappa^2}\int_{\partial\Sigma} N^{\mu\nu}\,\mathrm{d}\Sigma_{\mu\nu} &= \frac{4}{\kappa^2}\int_{S^7 \text{ at } r\to\infty} N^{r0}r^7\,\mathrm{d}\Omega_7 \\ &= \frac{6\omega_7}{\kappa^2}\epsilon_0'^{+}(M + Z\Gamma_0\Gamma_1)\epsilon_0\,, \end{aligned} \tag{5.6}$$

ω_7 being the area of the unit seven-sphere. On the other hand, using the identity

$$\overline{\hat{D}_\mu \epsilon'}\gamma^{\mu 0 \rho}\hat{D}_\rho \epsilon = \left(\hat{D}_k \epsilon'\right)^+ (\hat{D}^k \epsilon) - \left(\gamma^j \hat{D}_j \epsilon'\right)^+ \left(\gamma^k \hat{D}_k \epsilon\right) \tag{5.7}$$

and the field equations $G^{\mu\nu} = \kappa^2 T^{\mu\nu}$ and $\nabla_\rho \, \mathrm{e}^{-\phi} H^{\rho\mu\nu} = 0$ we have

$$\int_\Sigma \nabla_\mu N^{\mu\nu} \, \mathrm{d}\Sigma_\nu = \int \sqrt{-g_{00} h} \, \mathrm{d}^8 x \Big\{ \left(\gamma^j \hat{D}_j \epsilon'\right)^+ \left(\gamma^k \hat{D}_k \epsilon\right) - \left(\hat{D}_k \epsilon'\right)^+ (\hat{D}^k \epsilon) - \kappa^2 (\delta_{\epsilon'} \lambda)^+ (\delta_\epsilon \lambda) \Big\}, \tag{5.8}$$

where h_{ij} is the metric on any spacelike surface (not necessarily at infinity). Using eqs. (5.2), (5.6), and (5.8), and taking $\epsilon = \epsilon'$ we obtain

$$\frac{6\omega_7}{\kappa^2} \epsilon_0^+ (M + Z\Gamma_0 \Gamma_1) \epsilon_0 = \frac{8}{\kappa^2} \int \sqrt{-g_{00} h} \, \mathrm{d}^8 x \Big\{ \kappa^2 (\delta_\epsilon \lambda)^+ (\delta_\epsilon \lambda) + \left(\hat{D}_k \epsilon\right)^+ (\hat{D}^k \epsilon) - \left(\gamma^j \hat{D}_j \epsilon\right)^+ \left(\gamma^k \hat{D}_k \epsilon\right) \Big\}. \tag{5.9}$$

The first two terms on the right-hand side are nonnegative for commuting spinors, and the third one vanishes if the Witten condition [24] is satisfied, i.e.

$$\gamma^j \hat{D}_j \epsilon = 0 . \tag{5.10}$$

We thus have the inequality

$$\frac{6\omega_7}{\kappa^2} \epsilon_0^+ (M + Z\Gamma_0 \Gamma_1) \epsilon_0 \geqslant 0 , \tag{5.11}$$

provided eq. (5.10) holds. Using the fact that $\Gamma_0 \Gamma_1$ has eigenvalues ± 1 we conclude that if condition (5.10) is satisfied then

$$M \geqslant |Z| , \quad \text{and}$$

$$M = |Z| \quad \Leftrightarrow \quad \hat{D}_j \epsilon = 0 , \tag{5.12}$$

which is a Bogomol'nyi-type inequality. The central charge T is proportional to Z and is determined by the behaviour of the solutions at infinity, as in standard supersymmetry. In fact, Z is given, up to a constant, by the integral of the dual of the three-form H of eq. (5.5d) over a seven-sphere at infinity,

$$\int_{S^7 \text{ at } r \to \infty} {}^*H = Z\omega_7 . \tag{5.13}$$

Its absolute value $|Z|$ gives the Bogomol'nyi lower bound for the mass.

It is important to understand that Z, or equivalently T is not the central charge appearing in ordinary N-extended supersymmetry, but something else. To see this, let us look at the commutator of two supercharges associated to two supersymmetry transformations of parameters ϵ and ϵ':

$$\{Q^i[\epsilon], Q^j[\epsilon']\} = 2\delta^{ij}\bar{\epsilon}'\Gamma^a P_a \epsilon + \bar{\epsilon}' U^{ij}\epsilon. \tag{5.14}$$

Here $i, j = 1, 2, \dots, N$, P_a is the momentum and U^{ij} is the matrix of central charges, which is skew-symmetric, $U^{ij} = -U^{ji}$. Now, the solutions of the $N = 1$, $D = 10$ supergravity equations that we are considering are flat at spatial infinity, so an anticommutation relation of type (5.14) holds at $r \to \infty$. P_a then has to be interpreted as momentum per unit length at spatial infinity, or in other words as the ADM momentum per unit length. Since the metric (5.5a–c) is static, the only nonvanishing component of the ADM momentum P_a is the zero-component or energy, which as shown in sect. 3 coincides with the string tension and is given by $3M\omega_7/\kappa^2$. This, together with the fact that $U = 0$ for $N = 1, D = 10$, leads to

$$\{Q[\epsilon], Q[\epsilon']\} = 6(\omega_7/\kappa^2) M \epsilon_0'^{+} \epsilon_0, \tag{5.15}$$

where $\epsilon \to \epsilon_0$ and $\epsilon' \to \epsilon_0'$ as $r \to \infty$.

Now, from eqs. (5.2) and (5.6) it follows that

$$\delta_\epsilon Q[\epsilon'] = \{Q[\epsilon], Q[\epsilon']\} = 6(\omega_7/\kappa^2)\epsilon_0'^{+}(M + Z\Gamma_0\Gamma_1)\epsilon_0, \tag{5.16}$$

which does not look like the usual $D = 10, N = 1$ supersymmetry algebra. The reason is that $Q[\epsilon]$ is not the supercharge but the supercharge per unit length. If we dimensionally reduce to nine dimensions then we have $N = 1$ supersymmetry and $Q[\epsilon]$ is like the usual supercharge. We can rewrite eq. (5.16) as

$$\{Q[\epsilon], Q[\epsilon']\} = 2\bar{\epsilon}'\Gamma^a P_a \epsilon + \bar{\epsilon}' T \epsilon. \tag{5.17}$$

This is the $D = 9, N = 1$ supersymmetry algebra with a central charge $T = 6Z\omega_7/\kappa^2$ that is responsible for the partial breaking of supersymmetry. Note that in nine dimensions even the $N = 1$ supersymmetry algebra admits a central charge because the Γ_0 matrix is symmetric in the Majorana representation.

If we take a supersymmetric solution and set $\epsilon = \epsilon'$ then $\delta_\epsilon Q[\epsilon]$ vanishes, which from eq. (5.16) implies that

$$\epsilon_0^{+}(M + Z\Gamma_0\Gamma_1)\epsilon_0 = 0. \tag{5.18}$$

This is the condition for bosonic solutions with behaviour at spatial infinity given by eq. (5.5) to be supersymmetric. If Z is zero there is no such supersymmetric bosonic background since then the only solution to eq. (5.18) is $\epsilon_0 = 0$. For nonzero central charge eq. (5.18) has nontrivial solutions which correspond to partially broken supersymmetry. A three-form H with the fall-off at infinity of eq. (5.5d) is typically produced by a cosmic string source along the x^1 axis of the type discussed in sects. 2 and 3. The preceding argument reveals then the source as the cause of the partial breaking of supersymmetry.

Eq. (5.12) tells us that the Bogomol'nyi bound is saturated by supersymmetric solutions. A nice illustration of this is provided by the cosmic string background of sect. 3. This solution has $M = Z$ and in sect. 4 was proved to be supersymmetric. However, one still has to check that the string source does not contribute to the right-hand side of the Bogomol'nyi inequality via the field equations,

$$\frac{8}{\kappa^2}\int_{\mathrm{M}} \left(\nabla_\mu N^{\mu\nu}\right)_{\text{source}} \mathrm{d}\Sigma_\nu = 0\,. \tag{5.19}$$

Using (3.4a–c) for the last two terms on the right-hand side of eq. (5.4) and the expression for the energy–momentum tensor of the source, the left-hand side of eq. (5.19) can be written as

$$4\mu\int \mathrm{d}^8x\, \mathrm{e}^{3\phi_s/4}\,\delta^{(8)}(x)\,\bar{\epsilon}(1+\Gamma_0\Gamma_1)\epsilon\,, \tag{5.20}$$

that reproduces the condition (4.7) for supersymmetry, so eq. (5.19) holds.

6. Discussion

We have seen that a low-energy analysis of the structure of marcroscopic superstrings reveals a rich structure with many parallels in the theory of supersymmetric solitons. Although we have worked only to lowest order in α', the nonrenormalization of the mass of these states discussed in ref. [15] and in sect. 2 suggests that these features will persist to all orders in α'. We presume that there exist superconformal field theories which describe exactly the solutions that we have analyzed here in the low-energy approximation. It would be interesting to know how the existence of exact multi-string solutions and the Bogomol'nyi bound are manifested in the properties of these superconformal field theories. Just as the self-dual Yang–Mills equations and the Bogomol'nyi equations have a rich mathematical theory associated with their solutions, involving topics as diverse as twistors and integrable systems, we may expect that the construction of superconformal field theories corresponding to exact multi-string solutions of superstring theory will also reveal rich new mathematical structures.

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Appendix A

THE TWO-POINT FUNCTION AND MASS SHIFT

In this appendix we explicitly evaluate the two-point function which gives the mass shift for the macroscopic string states discussed in sect. 2 and show that it vanishes at one-loop level. The amplitude is given by

$$A(\zeta_i, k_i, \bar{k}_i) = \sum_{(\alpha,\beta)} \int_{\mathscr{M}_g} \frac{\mathrm{d(WP)}}{\mathrm{vol}(\ker P_1)} \mathrm{d}^2\nu_1\, \mathrm{d}^2\nu_2 \cdot Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) \cdot \bar{Z}(\bar{\tau})$$
$$\times \langle V(k_1, \zeta_1, \nu_1; \bar{k}_1, \bar{\zeta}_1, \bar{\nu}_1) V(k_2, \zeta_2, \nu_2; \bar{k}_2, \bar{\zeta}_2, \bar{\nu}_2)\rangle \qquad \text{(A.1)}$$

Here d(WP) is the Weil–Petersson measure over the genus-g moduli space $\mathscr{M}_g$ and we have summed over the spin structures (α, β). $Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau)$ and $\bar{Z}(\bar{\tau})$ are the partition functions in the right and the left sectors respectively. $\langle VV\rangle$ denotes the correlation function of the vertex operators for a given point in the moduli space obtained by integrating over only the matter fields.

The various correlations on a Riemann surface of genus g can be readily evaluated. The fermion propagator is given by the Sžego Kernel

$$\langle \psi^\mu(\nu_1)\psi^\nu(\nu_2)\rangle_{(\alpha,\beta)} = \eta^{\mu\nu} S\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu_{12}|\tau),$$

$$S\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu_{12}|\tau) = \frac{1}{E(\nu_{12}|\tau)} \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu_{12}|\tau)}{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau)}. \qquad \text{(A.2)}$$

where τ is the period matrix, $E(\nu|\tau)$ is the prime form and $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\nu|\tau)$ is the usual ϑ function with characteristics.

The bosonic part of the vertex operator looks very similar to that for a gauge boson, except that for a winding state $k \neq \bar{k}$. This means the path integral over $X(\nu, \bar{\nu})$ should be performed over a Riemann surface with a branch cut running from $(\nu_1, \bar{\nu}_1)$ to $(\nu_2, \bar{\nu}_2)$. We should integrate over the field configurations with winding number $\frac{1}{2}(k_i - \bar{k}_i)$ around the point $(\nu_i, \bar{\nu}_i)$. A quicker way to evaluate the correlations is to use factorization of higher genus partition function [25]. The partition function for the variable X^1 on a genus-g surface Σ_g is given by

$$Z(R) = Z^{\mathrm{cl}}(R) Z^{\mathrm{qu}},$$

$$Z^{\mathrm{cl}}(R) = \sum_{(p,\bar{p})\in\Gamma_R^g} \exp\{i\pi(p\cdot\tau\cdot p - \bar{p}\cdot\bar{\tau}\cdot\bar{p})\},$$

$$Z^{\mathrm{qu}} = \left(\frac{8\pi^2 \det' \Delta_g}{\int \mathrm{d}^2\nu \sqrt{g}}\right). \qquad \text{(A.3)}$$

Here R is the radius of the circle on which X^1 lives, p and $\bar{p}$ are the left and right-moving momenta running through the loops and lie on the lattice Γ_{R}^g, Δ_g is the scalar laplacian on the surface Σ_g.

By considering the partition function on a degenerate Riemann surface where an appropriate number of handles pinch off we can obtain the n-point function A_n on a lower genus Riemann surface:

$$A_n(k_i,\bar{k}_i;\nu_i,\bar{\nu}_i) = \left\langle \prod_{i=1}^{n} \mathrm{e}^{ik_i\cdot X^1(\nu_i)}\,\mathrm{e}^{i\bar{k}_i\cdot \bar{X}^1(\bar{\nu}_i)} \right\rangle$$

$$= \prod_{i<j} E(\nu_{ij}|\tau)^{k_i\cdot k_j}\bar{E}(\bar{\nu}_{ij}|\bar{\tau})^{\bar{k}_i\cdot\bar{k}_j}\vartheta\begin{bmatrix} k_i \\ \bar{k}_i \end{bmatrix}(\nu_i,\bar{\nu}_i|\tau,\bar{\tau}),$$

$$\vartheta\begin{bmatrix} k_i \\ \bar{k}_i \end{bmatrix}(\nu_i,\bar{\nu}_i|\tau,\bar{\tau}) = \frac{1}{Z^{\mathrm{cl}}}\sum \exp\Big\{ i\pi[(p\cdot\tau\cdot p)-(\bar{p}\cdot\bar{\tau}\cdot\bar{p})]$$

$$+2\pi i\left[\left(p\cdot\sum k_i\int^{\nu_i}\omega\right)-\left(\bar{p}\cdot\sum\bar{k}_i\int^{\bar{\nu}_i}\bar{\omega}\right)\right]\Big\}, \quad \text{(A.4)}$$

where $\omega_i(z)$ are the abelian differentials. For the noncompact dimensions we have the usual result

$$\left\langle \prod_{i,j} \mathrm{e}^{ik_i\cdot X(\nu_i,\bar{\nu}_i)} \right\rangle = \left| \prod_{i<j} F(\nu_{ij}|\tau)^{k_i\cdot k_j} \right|^2,$$

$$F(\nu_{ij}|\tau) = E(\nu_{ij}|\tau)\exp\left\{-\pi\,\mathrm{Im}\int_{\nu_i}^{\nu_j}\omega(\mathrm{Im}\,\tau)^{-1}\,\mathrm{Im}\int_{\nu_i}^{\nu_j}\omega\right\}. \quad \text{(A.5)}$$

With these correlations we are ready to evaluate the amplitude. For concreteness, let us consider the one-loop amplitude. The Weil–Petersson measure for the torus is $\mathrm{d}^2\tau/(\mathrm{Im}\,\tau)^2$ where the integration in τ is over the fundamental region $|\tau| \geqslant 1$, $\mathrm{Im}\,\tau > 0$ and $-\frac{1}{2} \leqslant \mathrm{Re}\,\tau \leqslant \frac{1}{2}$. The integration in ν_i is over the strip $0 \leqslant \mathrm{Im}\,\nu_i \leqslant \mathrm{Im}\,\tau$ and $-\frac{1}{2} \leqslant \mathrm{Re}\,\tau \leqslant \frac{1}{2}$. We can cancel vol(ker P_1) against $\int \mathrm{d}^2((\nu_1+\nu_2)/2)$ and set $\nu_2 = 0$ by using the single conformal Killing vector on the torus. Substituting these in eq. (A.1) we get the following result for the spin-structure-dependent part

$$A(\zeta_i,k_i,\bar{k}_i)|_{\text{torus}} = \sum_{(\alpha,\beta)}\int\frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2}\cdot Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\cdot\bar{Z}(\bar{\tau})\int\mathrm{d}^2\nu_{12}\,E(\nu_{12}|\tau)^{k_1\cdot k_2}$$

$$\times\bar{E}(\bar{\nu}_{12}|\bar{\tau})^{\bar{k}_1\cdot\bar{k}_2}\left(S\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu_{12}|\tau)\right)^2(\zeta_1\cdot k_2\zeta_2\cdot k_1-\zeta_1\cdot\zeta_2 k_1\cdot k_2)$$

$$\times\exp\left\{\frac{2\pi(\mathrm{Im}\,\nu_{12})^2}{\mathrm{Im}\,\tau}E_1E_2\right\}\vartheta\begin{bmatrix}k_1, & k_2\\ \bar{k}_1, & \bar{k}_2\end{bmatrix}(\nu_{12},\bar{\nu}_{12}|\tau,\bar{\tau}). \quad \text{(A.6)}$$

To see if singularities near $\nu_{12}=0$ arise in our case we can evaluate the function $\vartheta\begin{bmatrix} k_1, & k_2 \\ \bar{k}_1, & \bar{k}_2 \end{bmatrix}(\nu_{12}, \bar{\nu}_{12}|\tau, \bar{\tau})$ in the limit $R \to \infty$ by turning the sum into an integral:

$$\vartheta\begin{bmatrix} k_1, & k_2 \\ \bar{k}_1, & \bar{k}_2 \end{bmatrix}(\nu_{12}, \bar{\nu}_{12}|\tau, \bar{\tau}) \approx \exp\left\{ \frac{2\pi}{\operatorname{Im}\tau}\left[\frac{(\operatorname{Im}\nu_{12})^2}{R^2} - \frac{(\operatorname{Re}\nu_{12})^2 R^2}{4} - i \operatorname{Im}\nu \operatorname{Re}\nu \right]\right\}. \tag{A.7}$$

Let us look at the ν_{12} integral. In order to analytically continue the integrand in $k_1 \cdot k_2$ we take $(k_1 + k_2) = p$ and let $p^2/2 = k_1 \cdot k_2$ tend to zero only at the end of the calculation. We cannot continue $\bar{k}_1 \cdot \bar{k}_2$ independently because the total winding number must be conserved to satisfy the geometric requirement that the Riemann surface is properly branched. This means $(\bar{k}_1 + \bar{k}_2) = p$, i.e. $\bar{k}_1 \cdot \bar{k}_2 = (k_1 \cdot k_2 - 2)$. We can also use the following expansions in the region $\nu_{12} \ll 1$

$$E(\nu_{12}|\tau)^{k_1 \cdot k_2} \approx (\nu_{12})^{k_1 \cdot k_2}, \qquad S\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu_{12}|\tau) \approx \frac{1}{\nu_{12}} + \mathrm{O}(\nu_{12}). \tag{A.8}$$

Putting it all together we get the following expression for the ν_{12} integral:

$$\int_{|\nu_{12}|} \mathrm{d}^2\nu_{12}\, |E(\nu_{12}|\tau)|^{2k_1 \cdot k_2} \left(S\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu_{12}|\tau)\right)^2 \frac{1}{(E(\bar{\nu}_{12}|\tau))^2}$$

$$\times \exp\left\{ -\frac{\pi|\nu_{12}|^2 R^2}{2\operatorname{Im}\tau} - \frac{2\pi i \operatorname{Im}\nu_{12} \operatorname{Re}\nu_{12}}{\operatorname{Im}\tau}\right\}, \tag{A.9}$$

$$\approx \int_0^{2\operatorname{Im}\tau/\pi R^2} \rho\, \mathrm{d}\rho \int_0^{\pi} \mathrm{d}\theta\, \rho^{2k_i \cdot k_2} \left\{ \frac{1}{\rho^2 \mathrm{e}^{2i\theta}} + C\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right\}\left\{ \frac{1}{\rho^2 \mathrm{e}^{-2i\theta}} + \mathrm{O}(1)\right\}. \tag{A.10}$$

By performing the integral we see the spin-dependent part approaches a finite value as $k_1 \cdot k_2 \to 0$. Since the kinematic factor goes to zero as $k_1 \cdot k_2 \to 0$, the two-point function vanishes.

We expect that this argument would generalize to higher-genus surfaces. Any possible singularities can come only from the region $|\nu_{12}| \to 0$, i.e. from short distances on the world-sheet. This will be insensitive to the topology, hence independent of the genus.

Thus, the mass shift of these macroscopic string states should vanish identically to all order in perturbation theory.

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