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THE CHERN–SIMONS TERM VERSUS THE MONOPOLE

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Monopoles lead to linear confinement of electric charge in compact (2 + 1)-dimensional QED. On the other hand, the addition of a Chern–Simons term renders the Coulomb interaction short-range. We consider here continuum, compact QED with a Chern–Simons term and show that integrating over a collective coordinate associated with large gauge transformations cancels the monopole contribution to the partition function (or, more accurately, leads to confinement of monopole–antimonopole pairs). Thus compact (2 + 1)-dimensional QED with a Chern–Simons term can have deconfined electric charges with fractional statistics as predicted classically. If also true on the lattice, this result has crucial implications for some theories of high-T_c superconductors.

1. Introduction

The Coulomb interaction is logarithmic in two dimensions. Thus, one might expect (2 + 1)-dimensional QED to be confining with logarithmic interactions between charges, at least for weak coupling. Of course, since the coupling constant e^2 has dimensions of mass in (2 + 1) dimensions, weak coupling means $e^2 \ll m$, where m is the smallest matter field mass. (Different behaviour may occur at strong coupling. It has been recently suggested [1] that the theory with N flavours of massless fermions is not confining for large enough N.)

In fact, one must distinguish carefully between non-compact and compact QED. The theory can be rendered compact by obtaining the U(1) gauge group from the spontaneous breakdown of a compact non-abelian group (the simplest example being SO(3)). Alternatively, one may stick with the original U(1) gauge group but define the model on a lattice, with the gauge fields being the phases of link variables. Either way, there are monopoles [2] with Coulomb interactions. These appear as instantons in (2 + 1) dimensions, i.e. finite action solutions in euclidean

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space. Debye screening in the monopole gas implies an area law for the Wilson loop and hence linear confinement [3]. This occurs for arbitrarily weak coupling, but the length scale at which the potential crosses over from logarithmic to linear becomes exponentially large at weak coupling.

It is possible to add a Chern–Simons term [4] to the lagrangian. This gives the photon a mass, rendering the Coulomb interaction short-range, and thus destroying confinement in the classical approximation. Furthermore, the deconfined charged particles obtain fractional statistics [5]. Such particles are the basis of a popular theory of high-T_c superconductors [6]. Indeed, the $(2+1)$-dimensional Hubbard model, used to describe the Cu–O_2 layers in high-T_c materials, can be mapped approximately [7] onto QED, and, in phases where parity and time reversal are spontaneously broken, naturally develops a Chern–Simons term [8].

However, due to the underlying lattice structure, the approximate mapping gives *compact* QED. Thus it is important to understand the interplay between monopoles and the Chern–Simons term. They clearly tend to work in opposite directions since the former promotes the confining interaction between electric charges from logarithmic to linear while the latter makes it drop off exponentially.

One's first thought is that the Chern–Simons term should be allowed to alter the classical monopole solution, such that the interaction between monopoles becomes short-range. However, this approach opens up a variety of unpleasant issues. In particular, since the Chern–Simons term is imaginary in euclidean space, including it in the equations of motion necessarily makes the classical solution complex. We have not been able to find appropriate solutions of these complex equations and suspect that none may exist. An alternative approach is to consider the usual monopole solution and treat the Chern–Simons term at the one-loop level. This is possible since the Chern–Simons term is finite (though gauge-dependent) when evaluated for a monopole. It is in fact the sensible approach if the dimensionless Chern–Simons term is O(1) while $e^2 \ll m$.

The continuum lagrangian density takes the form

$$\mathscr{L} = (1/4e^2)F^2 + if\varepsilon_{\mu\nu\lambda}A_\mu F_{\nu\lambda},$$

in the abelian case. In the compact case, A_μ is to be thought of as a phase variable so that A_μ is equivalent to $A_\mu + 2\pi/a$ where a is the lattice spacing. Therefore, this continuum form of the Chern–Simons term can only be correct for small A_μ. The form of the Chern–Simons term in the compact lattice theory is not very well understood. In the non-abelian case, for an SO(3) gauge group, the gauge field lagrangian becomes

$$\mathscr{L}_G = (1/4e^2)F^aF^a + if\varepsilon_{\mu\nu\lambda}\left(A_\mu^a\partial_\nu A_\lambda^a + \tfrac{2}{3}\varepsilon^{abc}A_\mu^a A_\nu^b A_\lambda^c\right)$$

and there is an additional Higgs-field sector of the lagrangian

$$\mathscr{L}_{\mathrm{H}} = \tfrac{1}{2}\left(D_\mu\varphi^a\right)\left(D_\mu\varphi^a\right) - \tfrac{1}{2}m^2\varphi^a\varphi^a + \tfrac{1}{4}\lambda\left(\varphi^a\varphi^a\right)^2.$$

In addition, we may have some charged massive matter fields, fermions in the case of the high-T_c inspired model. After the Higgs mechanism takes place, the only low energy particles (at scales below the Higgs mass, m_{H} and the W boson mass, m_{W}) are the photon, plus possible additional light matter fields.

The dilute monopole gas approximation is justified in the weak-coupling limit. We are primarily interested in the case where f is O(1). In the non-abelian theory invariance under compact gauge transformations quantizes f [9]: $f = n/8\pi$ where n is an integer. (This result is reviewed below.) In some versions of the high-T_c inspired model, the Chern–Simons term arises at the one-loop level from integrating out charged fermions. In this case, for the non-abelian theory, we obtain the above form for f. In the abelian case we obtain $f = n/8\pi$ from integrating out fermions. Since the number of fermions is generally even because of the usual lattice doubling, this becomes $f = n/4\pi$. We suspect that gauge invariance may again impose this quantization of f in the lattice theory but we do not attempt to give a proof of this here. In the high-T_c inspired models $f = n/4\pi$ for the SU(n) Hubbard model [$1/2\pi$ for the realistic SU(2) case]. Thus we are primarily interested in the case where f is O(1). Unfortunately, e^2/m may also be O(1) in the high-T_c inspired models, but we may, at least as a warm-up, consider the weak-coupling limit. Thus it is natural to treat the Chern–Simons term at the one-loop level. This is especially so in the case where it arises from the phase of a fermion determinant. Thus we consider the Chern–Simons term when we integrate over fluctuations around the monopole but not in determining the classical monopole solution.

Since the Chern–Simons term is pure imaginary, it is possible that some part of the functional integral "in the vicinity of a monopole" may average this phase over 2π and make the monopole contribution to the partition function vanish. We will argue that this is, in fact, the case. We can make this rather vague idea precise by considering a particular collective coordinate for the monopole solution which leaves the real part of the action invariant but shifts the imaginary part. This coordinate labels a non-compact gauge transformation. In the next two sections we discuss the single-monopole sector: the abelian lattice case in sect. 2 and the non-abelian continuum case in sect. 3. In fact, gauge invariance of the Chern–Simons term requires that the theory be defined with boundary conditions which forbid both a net magnetic charge *and* non-compact gauge transformations (i.e. those which do not approach constants at ∞). However, the results of sects. 2 and 3 *can* be applied to an isolated monopole in a dilute gas of net magnetic charge 0. This, and other points are dealt with in sect. 4.

2. Abelian case

As a warm-up consider the abelian case. We might make a gauge transformation on the monopole

$$A_\mu \to A_\mu + \partial_\mu \Lambda\,,$$

such that Λ does not vanish at infinity, but rather winds through one period,

$$\Lambda(x=0)=0\,,\qquad \Lambda(x\to\infty)=2\pi\,.$$

Note that a gauge transformation of this type is consistent with any imposed boundary conditions at $x=\infty$ since $\Lambda=2\pi$ is equivalent to $\Lambda=0$ in *compact* QED. How does the Chern–Simons term change under this gauge transformation? Naively, we obtain

$$\delta S_{\mathrm{CS}} = if\int(\partial_\mu\Lambda)B_\mu\,.$$

Here we adopt the notation $F_{\mu\nu}\equiv\varepsilon_{\mu\nu\lambda}B_\lambda$. (Of course, in (2 + 1) dimensions only B_0 is actually the magnetic field; B_1 and B_2 are the electric fields.) Integrating by parts and using the Bianchi identity, $\partial_\mu B_\mu=0$, we obtain

$$\delta S_{\mathrm{CS}} = if\int \mathrm{d}S_\mu\,\Lambda B_\mu = if\cdot 2\pi\Phi\,,$$

where Φ is the magnetic flux of the monopole. In compact QED this flux must be $2\pi n$ where n is an integer in order that the string threading between the plaquettes of the lattice can be removed by a 2π gauge transformation. Thus we see that the exponentiated action is only invariant under gauge transformations which vanish at infinity if $f=n/2\pi$ (n integer). However, monopoles are not consistent with any fixed or periodic boundary conditions at ∞ (for this reason the boundary conditions usually require them to appear in pairs; see the discussion below). Thus we should relax the condition on Λ and let it be an arbitrary constant at ∞. We then find

$$\delta S_{\mathrm{CS}} = if\Lambda\cdot 2\pi\,,$$

for a flux 2π monopole. There is no reason to consider any particular value of Λ rather than any other in defining the classical monopole configuration; they all lead to the same behaviour of A_μ at infinity. Integrating over Λ as a collective coordinate and using the fact that the real part of the action is invariant even under non-compact gauge transformations, we obtain a factor in the partition function of

$$\int_0^{2\pi}\frac{\mathrm{d}\Lambda}{2\pi}\,\mathrm{e}^{if\Lambda 2\pi}\,.$$

Setting $f = n/2\pi$, this integral gives $\delta_{n,0}$. Thus the contribution of the monopole is wiped out by integrating over this collective coordinate.

In fact, there are several points about the above argument which are unsatisfactory. The naive continuum form for the Chern–Simons terms used above, cannot be precisely correct in a compact lattice theory since it should be invariant under shifting A_μ by $2\pi/a$. (While we might have the bad taste to simply *define* our Chern–Simons term this way, it is certainly not what would be obtained from integrating our fermions coupled to a compact gauge field.) Also the Bianchi identity $\partial_\mu B_\mu = 0$ may not be satisfied in the compact lattice theory; it depends on exactly what we mean by B_μ. In any event, if we think of B_μ as the naive continuum Dirac field which *does* obey the Bianchi identity, then there will be a string penetrating the sphere at ∞ which precisely cancels the spherically symmetric part of the magnetic flux so that $\delta S_{\mathrm{CS}} = 0$. Thus this argument is not really clean enough to show that f is quantized as $n/2\pi$; indeed we can apparently obtain $f = 1/4\pi$ from integrating out the minimal number of fermions (two due to lattice doubling in the continuum time formulation). The argument is also not clean enough to show that there is a collective coordinate integration which wipes out the monopole. We are actively considering these issues. In this paper we content ourselves with working out the non-abelian case which seems much more straightforward due to the absence of the ambiguities introduced by the lattice.

3. Non-abelian case

We consider the monopole in the SO(3) gauge theory, spontaneously broken to U(1) by a triplet Higgs field, as written above. The 't Hooft–Polyakov monopole can be written in a spherically symmetric gauge in which $\boldsymbol{\varphi} = \varphi(r)\hat{\boldsymbol{r}}$. Here $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction and, at ∞, $\varphi(r) \to \varphi_0$, the Higgs expectation value. $\varphi(r)$ must vanish at the origin to avoid a singularity. It reaches its asymptotic value at $r \approx 1/m_{\mathrm{H}}$, the inverse Higgs mass. The triple gauge field takes the form

$$A_\mu^a = \varepsilon_{a\mu i}\hat{r}_i[1 - K(r)]\big/r, \tag{1}$$

where $K(r)$ goes to 0 at $r = \infty$ and to 1 at $r = 0$. It reaches its asymptotic value at distances of order $r \approx 1/m_{\mathrm{W}}$. The non-abelian field strength is

$$B_\mu^a = \hat{r}_\mu\hat{r}_a\frac{(1-K^2)}{r^2} + \frac{(\hat{r}_\mu\hat{r}_a - \delta_{\mu a})}{r}\frac{\mathrm{d}K}{\mathrm{d}r} \to \frac{\hat{r}_\mu\hat{r}_a}{r^2}\,.$$

We may define the abelian magnetic field as $B_\mu^a\hat{\varphi}^a$. This indeed has the standard monopole form with a flux of 4π, twice the Dirac value.

At this point we may immediately display our collective coordinate. We make a gauge transformation of the form

$$g = \exp[i\Lambda(r)\hat{\boldsymbol{r}}\cdot\boldsymbol{\sigma}/2]. \tag{2}$$

We will require $\Lambda(0)=0$ to avoid a singularity, and we will allow $\Lambda(\infty)$ to be a non-zero constant, Λ. The detailed form of $\Lambda(r)$ will not matter. Different choices of $\Lambda(r)$ with the same asymptotic value, Λ, are related by *compact* gauge transformations. Note that $\Lambda = 2\pi$ corresponds to the identity transformation on our adjoint representation fields. This gauge transformation leaves $\boldsymbol{\varphi}$ unchanged and transforms A^a_μ into

$$A^a_\mu = \varepsilon_{a\mu i}\hat{r}_i \frac{1-K\cos\Lambda}{r} + \frac{(\delta_{a\mu}-\hat{r}_\mu\hat{r}_a)}{r}K\sin\Lambda + \hat{r}_\mu\hat{r}_a\frac{\mathrm{d}\Lambda}{\mathrm{d}r}.$$

For $r \gg m_\mathrm{W}^{-1}$, m_H^{-1},

$$A^a_\mu \to \varepsilon_{a\mu i}\hat{r}_i/r + \hat{r}_\mu\hat{r}_a\frac{\mathrm{d}\Lambda}{\mathrm{d}r}.$$

Thus this gauge transformation acts asymptotically only on the "abelian component" of A^a_μ, $\hat{\varphi}^a A^a_\mu$. Since Λ goes to a constant at ∞, the ultimate asymptotic behaviour of A^a_μ is unaffected by the gauge transformation. Therefore, there is no reason not to average over all functions $\Lambda(r)$ in the monopole sector of the path integral. Indeed, a correct evaluation of the path integral requires that we do so, treating Λ as a collective coordinate.

Now let us consider the Chern–Simons term. It is easy to see that S_CS vanishes when $\Lambda = 0$ since it involves an antisymmetric triple product and both A^a_μ and its derivative involve only the single vector $\hat{\boldsymbol{r}}$. In general, for arbitrary A^a_μ, under a (non-infinitesimal) gauge transformation,

$$\delta S_\mathrm{CS} = 2if\varepsilon_{\mu\nu\lambda}\operatorname{tr}\left[-\tfrac{1}{3}\int \mathrm{d}^3\boldsymbol{x}\left(\partial_\mu g g^{-1}\right)\left(\partial_\nu g g^{-1}\right)\left(\partial_\lambda g g^{-1}\right) + \int \mathrm{d}S_\mu\, A_\nu \partial_\lambda g g^{-1}\right].$$

(Here, $A_\nu \equiv \frac{1}{2}A^a_\nu\sigma^a$, where the σ^a are Pauli matrices. The second term is a surface integral.) Without considering monopoles, we would normally require A_μ to vanish at ∞ so only the first term survives. If we assume that $g(\boldsymbol{x})$ is compact, i.e. independent of direction at $\boldsymbol{x}\to\infty$, then the first term is proportional to the jacobian of the mapping of the SU(2) group manifold, S^3, onto the physical space which is also equivalent to S^3, $\Pi^3(S^3) = Z$. Thus

$$\delta S_\mathrm{CS} = if\cdot 16\pi^2 n \qquad (n \text{ integer}).$$

Requiring invariance of $\exp[-S]$ under such a compact gauge transformation gives the well-known quantization of the Chern–Simons coupling:

$$f = n/8\pi \qquad (n \text{ integer}).$$

However, when a monopole is present the space cannot be compactified and A_μ does not vanish rapidly enough at ∞ for the second term to give zero. An explicit calculation shows, for A_μ and g given by eqs. (1) and (2):

$$\delta S_{\mathrm{CS}} = 2if \int \mathrm{d}^3\boldsymbol{x} (1/r^2)\, \mathrm{d}\Lambda/\mathrm{d}r = if \cdot 8\pi\Lambda .$$

Not surprisingly, since the gauge transformation is essentially abelian, δS_{CS} has essentially the abelian form discussed above. Note that there is no string to worry about in the non-abelian case. The final step in our argument is to integrate over the collective coordinate Λ from 0 to 2π. Assuming f has one of the allowed values, $n/8\pi$, this integration gives a factor of $\delta_{n,0}$.

4. Discussion

The collective coordinate, Λ, has actually appeared before in the physics literature, but in the context of monopoles as finite-energy configurations in the (3 + 1)-dimensional theory. In (3 + 1) dimensions this collective coordinate must be treated as a rotator variable. Periodic gauge transformations on the monopole of the form $g = \exp[i\omega t \Lambda(r) \hat{\boldsymbol{r}} \cdot \boldsymbol{\sigma}/2]$, (with A_0 still equal to zero) give the dyon solution [10]. Actually, in the dyon case, the detailed form of $\Lambda(r)$ is not arbitrary, but is determined precisely by the field equations. This occurs since although A_i^a is obtained by a time-dependent gauge transformation from the monopole configuration, A_0 is held fixed and thus the dyon is not a true gauge transformation of the monopole.

It is amusing to note (and possibly helpful for understanding the abelian lattice case) that we can choose the function $\Lambda(r)$ to be zero inside the monopole core and vary between 0 and Λ over distances large compared to the core size. In this case $\Lambda(r)$ is only non-zero in the region where the monopole looks abelian. The Chern–Simons term can be decomposed into abelian and non-abelian parts

$$S_{\mathrm{CS}} = if \int \mathrm{d}^3\boldsymbol{x} \left[A_\mu^a B_\mu^a - \tfrac{1}{3} \varepsilon_{\mu\nu\lambda} \varepsilon_{abc} A_\mu^a A_\mu^b A_\lambda^c \right].$$

Upon making the gauge transformation of (2), the first term simply gives

$$if \int \mathrm{d}^3\boldsymbol{x}\, \partial_\mu \Lambda\, B_\mu ,$$

where B_μ is the abelian magnetic field, $\hat{\varphi}^a B^a_\mu \approx \hat{r}_\mu / r^2$. Thus we obtain $if\Lambda\Phi = if \cdot 4\pi\Lambda$. Somewhat surprisingly, this is only half the full answer given above; the other half comes from the non-abelian part of the Chern–Simons term.

Actually, this discussion has so far been a bit cavalier about boundary conditions and the treatment of dilute multi-monopole configurations. The imposition of, for example, periodic boundary conditions will ensure that only configurations of net magnetic charge 0 occur. At weak coupling, the density of monopoles is $O(\exp[-\mathrm{const}/e^2])$. Thus the typical spacing between monopoles is much greater than their core size and the only significant interaction between them is the $1/r$ magnetic Coulomb interaction. Any other interaction depending on the structure of the monopole core should drop off faster than $1/r$. It is precisely this Coulomb interaction which produces Debye screening and hence area law behaviour for the Wilson loop. We will not discuss in detail here the optimal way of constructing multi-monopole configurations; it is presumably done by patching together the non-abelian parts of the fields and adding the abelian parts on domains much larger than the monopole core. Our collective coordinate, Λ, is compatible with such a multi-monopole construction because the Λ-dependence of the fields vanishes as $\mathrm{d}\Lambda/\mathrm{d}r$, which we can take to be zero outside some distance of order the monopole core size. Thus we can associate a collective coordinate, Λ, with each monopole in the dilute gas. Any Λ-dependent interaction between the monopoles should vanish exponentially at distances much larger than this. Therefore, in the dilute gas approximation, we have an independent integration over a collective coordinate, Λ, for each monopole in the gas and hence zero contribution to the partition function from monopoles.

In the higher order corrections to the dilute gas approximation there will be a contribution from monopoles but there is effectively a confining force between monopole–antimonopole pairs*. To see this, note that the action for a monopole–antimonopole pair separated by a distance $r \gg$ core size, ξ, with collective coordinates Λ_+, Λ_-, takes the form

$$S = in(\Lambda_+ - \Lambda_-) + \mathrm{e}^{-r/\xi} g(\Lambda_+ - \Lambda_-) + (\Lambda\text{-independent terms}),$$

where g is some function of the relative collective coordinate and comes from the real part of the action. g cannot depend on the average of the collective coordinates because this can be shifted by a gauge transformation which leaves the real part of the action invariant. We assume the gas to be dilute so that statistically important configurations have $r \gg \xi$. Therefore we may expand in powers of $\mathrm{e}^{-r/\xi}$ (corresponding to a virial expansion). We then obtain a factor in the partition function of the form

$$\mathrm{e}^{-r/\xi} \int \mathrm{d}\Lambda\, \mathrm{e}^{in\Lambda} g(\Lambda).$$

* We are indebted to L. Yaffe for pointing this out to us.

Thus the effect of integrating over the relative collective coordinate is to give a term in the effective action linear in the monopole–antimonopole separation. We expect this monopole gas to be in a confined phase and hence not to obtain an area law contribution from it to the Wilson loop for electric charges. The confined monopole–antimonopole gas should simply make small corrections to perturbation theory which do not change any of the qualitative behaviour of the model.

The above results may also have implications for the lattice CP^1 or O(3) non-linear σ-model with a Hopf topological term. Haldane has argued [11] that in the lattice version of this model, hedgehog configurations, where the unit vector field $\hat{\boldsymbol{n}}$ is everywhere radial, may be important. The O(3) hedgehog is equivalent to the U(1) monopole in the CP^1 formulation. This can be seen by observing that the field strength of the CP^1 model can be expressed in terms of $\hat{\boldsymbol{n}}$ as

$$F_{\mu\nu} \equiv \hat{\boldsymbol{n}} \cdot \left(\partial_\mu \hat{\boldsymbol{n}} x \partial_\nu \hat{\boldsymbol{n}} \right).$$

For a hedgehog, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$, $F_{\mu\nu} = \varepsilon_{\mu\nu\lambda}\hat{r}_\lambda / r^2$. The Hopf term becomes the Chern–Simons term in the CP^1 formulation. It is possible that a collective coordinate integration again wipes out the hedgehog/monopole contribution to the partition function, in the presence of the Hopf/Chern–Simons term.

Haldane argued that the hedgehog gas leads to ground state degeneracy in the O(3) model with the added Berry's phase terms that occur upon deriving the σ-model from a quantum antiferromagnet with half-odd integer or odd spin. The argument was extended to the case of monopoles in QED with a Berry's phase term (again obtained from a quantum antiferromagnet) by Read and Sachdev [12] who showed that the degeneracy corresponds to broken translation invariance. These results tend to suggest that the ground state of the two-dimensional antiferromagnet always has a dimerized character (when it is not Neel ordered). Such a state has a basically trivial localized excitation spectrum and seems unlikely to have anyonic excitations or to be related to superconductivity. The arguments of this paper suggest that in a phase of broken T and P in which a Hopf/Chern–Simons term is generated, the effects of hedgehogs/monopoles are cancelled and thus the translation symmetry may remain unbroken and anyonic excitations may occur.

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Note added

After completing this work we received a preprint from X.G. Wen and A. Zee, (I.T.P.) which also concludes that monopole–antimonopole pairs are linearly confined in a model for high-T_c superconductors. However, the Chern–Simons term did not seem to play any role in their argument.

Note added in proof

After completion of this work we became aware of a paper by Pisarski [13] where conclusions similar to ours are reached by studying infinite action classical monopole solutions to the field equations with the Chern–Simons term taken into account. It is interesting that the monopole confinement can proceed through a dynamical mechanism (the infinite action of the solutions) as well as the topological mechanism that we have found. The precise relationship between the two deserves further investigation.

References

[1] E. Dagotto and J.B. Kogut, preprint ILL-(TH)-88-30
[2] G. 't Hooft, Nucl. Phys. B79 (1974) 276;
A.M. Polyakov, JETP Letters 20 (1974) 194
[3] A.M. Polyakov, Phys. Lett. B59 (1975) 82; Nucl. Phys. B120 (1977) 429
[4] J. Schoenfeld, Nucl. Phys. B185 (1981) 157;
R. Jackiw and S. Templeton, Phys. Rev. D24 (1981) 2291
[5] F. Wilczek, Phys. Rev. Lett. 49 (1982) 957;
D. Arovas, J.R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. B251 (1985) 117
[6] R.B. Laughlin, Science 242 (1988) 525; Phys. Rev. Lett. 60 (1988) 2677
[7] R. Baskaran and P.W. Anderson, Phys. Rev. B37 (1988) 580;
I. Affleck and J.B. Marston, Phys. Rev. B37 (1988) 3774;
J.B. Marston and I. Affleck, Phys. Rev. B39 (1989) 11538
[8] X.G. Wen, F. Wilczek and A. Zee, Phys. Rev. B39 (1989) 11413
[9] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 475
[10] B. Julia and A. Zee, Phys. Rev. D11 (1975) 2227
[11] F.D.M. Haldane, Phys. Rev. Lett. 61 (1988) 1029
[12] N. Read and S. Sachdev, Phys. Rev. Lett. 62 (1989) 1694
[13] R. Pisarski, Phys. Rev. D34 (1986) 3851