

WORLD SHEET APPROACH TO HETEROTIC INSTANTONS AND SOLITONS

Curtis G. CALLAN, Jr.*

Department of Physics, Princeton University, Princeton, NJ 08544, USA

Jeffrey A. HARVEY**

Enrico Fermi Institute, University of Chicago, 5640 Ellis Avenue, Chicago, IL 60637, USA

Andrew STROMINGER***

Department of Physics, University of California, Santa Barbara, CA 93106, USA

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Instantons and soliton solutions of heterotic string theory are investigated with an emphasis on the worldsheet point of view. The solitons have the structure of a fivebrane in ten dimensions, the instantons are simply related to the solitons by Wick rotation. Some new fivebrane solutions are presented with a four-dimensional cross-section of the fivebrane given by an infinitely long, semi-wormhole at the core stabilized by axion charge and containing non-abelian gauge excitations. The generic such solution is shown to correspond to a sigma model with $(4,0)$ worldsheet supersymmetry. At special symmetric points in the moduli space of multi-instanton or multi-soliton solutions, the spin connection with torsion becomes identical to the Yang–Mills gauge connection. This results in $(4,4)$ worldsheet supersymmetry, and we argue that the resulting sigma models with torsion are finite and conformally invariant with no α' corrections. It is further shown that the wormhole throat is then described by an exactly soluble conformal field theory which is essentially a Wess–Zumino–Witten model whose level is related to the axion charge. Because of the simplicity and exactness of these symmetric solutions, they should provide a useful starting point for the analysis of such issues as integration of collective coordinates in string theory, duality between strings and fivebranes and the quantization of fundamental fivebranes.

1. Introduction

In view of the role that soliton and instanton solutions have played in the development of a non-perturbative understanding of gauge field theories, it is natural to look for similar solutions of string theory. There are two goals of such a

* Internet: cgc@pupthy.princeton.edu

** Internet: harvey@curie.uchicago.edu

*** Bitnet: andy@voodoo

search: first, and best, one might find solitons with distinctive physical properties (statistics, charge, etc.) that are not obtainable in ordinary field theory and whose detection would serve as a direct signal of the existence of strings; second, learning how to quantize soliton collective coordinates and integrate over instanton collective coordinates is a useful first step on the road to understanding the non-perturbative quantum physics of strings. By and large, previous work along these lines has been limited to finding solitonic solutions of the low-energy space-time effective action of various string theories and has not addressed essentially stringy questions such as the effect of higher-order in α' corrections to the space-time action and the properties of the underlying superconformal sigma model.

Recently some intriguing results have been obtained concerning soliton strings having the quantum numbers (and other properties) of the fundamental strings themselves [1,2] and fivebrane solitons (domain walls with $5+1$ dimensional translation symmetry) which may be dynamically dual to fundamental strings [3]. The latter possibility raises the issue of quantizing fundamental p -brane theories [4]: for higher than $1+1$ dimensional worldsheets, the analog of the Polyakov path integral is non-renormalizable and it is not obvious how to first-quantize such a theory. The fivebrane soliton, because it has a finite-size core, has a built-in cutoff, and learning how to quantize its collective coordinates should give us some hints as to whether or not the fundamental theory can be quantized in a consistent way. The fivebrane solutions can also be directly related (via Wick rotation) to four-dimensional generalizations of Yang–Mills instantons and wormholes and may therefore give rise to non-perturbative corrections to the low-energy effective action of string theory.

Our aim in this paper is to present some first steps in an exploration of the essentially stringy properties of the fivebrane/instanton solutions mentioned above. We are, in particular, interested in the properties of the underlying conformal field theory. In the course of this work, we have come across a particular class of solutions which, without any $O(\alpha')$ corrections, seem to be exact solutions of the heterotic string. This solution has a number of interesting features: the core of the fivebrane contains the gauge fields of an instanton and the metric of an infinitely long wormhole throat stabilized by axion charge. The spin connection with torsion is equated with the gauge connection (as in $(2,2)$ Calabi–Yau compactification), yielding an enhanced $(4,4)$ worldsheet supersymmetry. In addition, the string coupling constant becomes large in the throat of the wormhole. The conformal field theory describing the wormhole throat is the product of a supersymmetric Wess–Zumino–Witten model with a free superfield with a background charge and can be shown to have $(4,4)$ superconformal symmetry. Because this solution is an exact solution of string theory we expect it to play an important role in our understanding of semi-classical string theory. On the other hand, some of the space-time features of the solution (e.g. the nontrivial spatial topology and strongly coupled region of the wormhole throat) are not yet well understood and should

probably be analyzed directly from the point of view of superconformal field theory without any explicit reference to the space-time structure.

The sigma model underlying the octonionic soliton/instanton of [5] appears to have quite a different structure than those discussed here and will hopefully be separately discussed in future work. We mention that related work in ref. [7] uses low-energy space-time supersymmetry to argue that exact, degenerate five-brane/instanton solutions of string theory corresponding to Yang–Mills instantons exist for every instanton scale size. In ref. [7] the dynamics of a variety of fivebrane solitons are described in terms of low-energy effective worldbrane actions.

The plan of this paper is as follows: in sect. 2 we review the construction of the known fivebrane solitons as supersymmetric solutions of the space-time equations of motion. We then describe the symmetric solution and explain why we expect, from the space-time point of view, that it remains an exact solution to higher orders in α' . In sect. 3 we construct the sigma model underlying these solutions. It is shown that the generic solution has $(4, 0)$ worldsheet supersymmetry and that the symmetric solution has enhanced $(4, 4)$ supersymmetry, which is then shown to protect it from renormalization. In sect. 4 we examine in more detail the Wess–Zumino–Witten sigma model which describes the asymptotic geometry of the wormhole throat. In sect. 5 we present our conclusions and suggestions for further work. In appendix A we show that $c = 6$, $N = 4$ sigma models with torsion have a finite number of moduli (and describe the geometry in detail), a result needed for sect. 3.

2. Instantons and solitons in low-energy field theory

In this section we will discuss supersymmetric instanton and soliton solutions to the low-energy equations of motion of heterotic string theory. The soliton solutions describe objects with five spatial dimensions, or fivebranes, embedded in nine-dimensional space. The time evolution of these fivebranes determines a $5 + 1$ -dimensional worldbrane embedded in ten-dimensional space-time which is invariant under $SO(5, 1)$ Lorentz transformations. The instanton solutions are trivially obtained from these by Wick rotation, and describe the product of a non-trivial four-dimensional configuration with a flat six-dimensional euclidean space. In the context of string compactification, the flat six-dimensional space may be replaced with the appropriate $c = 9$ conformal field theory. One thereby obtains an instanton which contributes to non-perturbative four-dimensional physics.

2.1. THE GAUGE SOLUTION

The first solution of this form was given in ref. [3]. In this subsection we will review this solution and bring out certain features which provide motivation for

considering a new type of solution which will be described in the following subsections.

The starting point for the solution of ref. [3] is the low-energy heterotic string effective action for the bosonic fields in the $D = 10$, $N = 1$ supergravity and super Yang–Mills multiplets

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{g} e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{3}H^2 - \frac{1}{30}\alpha' \text{Tr} F^2 \right), \quad (2.1)$$

and the corresponding supersymmetry transformation laws for the fermionic fields

$$\begin{aligned} \delta\chi &= F_{MN}\gamma^{MN}\epsilon, \\ \delta\lambda &= (\gamma^M\partial_M\phi - \frac{1}{6}H_{MNP}\gamma^{MNP})\epsilon, \\ \delta\psi_M &= (\partial_M + \frac{1}{4}\Omega_{-M}^{AB}\Gamma_{AB})\epsilon. \end{aligned} \quad (2.2)$$

Here the generalized connection is given by

$$\Omega_{\pm M}^{AB} = \omega_M^{AB} \pm H_M^{AB}, \quad (2.3)$$

and

$$dH = \alpha'(\text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F). \quad (2.4)$$

The trace is conventionally normalized so that $\text{Tr} F \wedge F = \sum_i F^i \wedge F^i$ with i an adjoint index. Our conventions agree with those of ref. [3] except in the sign of ϕ .

Denote world indices of the four-dimensional space transverse to the fivebrane by $\mu, \nu = 6 \dots 9$ and the corresponding tangent space indices by m, n . If we assume that the only non-trivial fields entering into the solution are those with transverse indices we see that the gaugino and gravitino variations involve the linear combinations of space-time $\text{SO}(4)$ generators $F = F_{\mu\nu}\gamma^{\mu\nu}$ and $\Omega_{-\mu} = \Omega_{-\mu}^{mn}\Gamma_{mn}$. Half of the gaugino and gravitino variations in (2.2) will vanish as long as F and $\Omega_{-\mu}$ are in the same $\text{SU}(2)$ subgroup of $\text{SO}(4)$. This is equivalent to the duality conditions

$$F_{\mu\nu} = \pm \frac{1}{2}\epsilon_{\mu\nu}^{\lambda\rho}F_{\lambda\rho}, \quad (2.5)$$

$$\Omega_{-\mu}^{mn} = \pm \frac{1}{2}\epsilon^{mn}_{pq}\Omega_{-\mu}^{pq}. \quad (2.6)$$

The solution of ref. [3], which we shall refer to as the “gauge” solution, has as its starting point a solution to the self-dual or anti-self-dual Yang–Mills equation (2.5). For simplicity we henceforth consider only the self-dual case. The gauge fields are taken to be $\mathcal{O}(\alpha'^0)$. They induce corrections to the constant dilaton, flat metric and antisymmetric tensor field which are $\mathcal{O}(\alpha')$. (The Riemann curvature is

then $O(\alpha')$ which justifies neglect of curvature-squared terms in (2.1) to leading non-trivial order.)

We first discuss the gauge field configuration. In general, solutions to the self-dual $SU(2)$ Yang–Mills equations of topological charge k have a moduli space of dimension $8k - 3$ with the parameters corresponding physically to the locations and scale sizes of the k instantons ($5k$ parameters) and the relative $SU(2)$ orientations of the k instantons ($3k - 3$ parameters). For many purposes it is useful to also include the overall global $SU(2)$ orientation which results in a moduli space of dimension $8k$. The study of the full moduli space of multi-instanton solutions requires sophisticated mathematical machinery [8]. However there is a simple ansatz, due to 't Hooft [9], which captures a $5k$ -dimensional subspace of the full moduli space. In our later construction this ansatz will play an important role and we will see physically why it does not include the $SU(2)$ part of the moduli space.

The ansatz consists of writing

$$A_\mu = \bar{\Sigma}_{\mu\nu} \nabla^\nu \ln f, \quad (2.7)$$

with $\bar{\Sigma}_{\mu\nu}$ antisymmetric and anti-self-dual. Substituting this ansatz into the self-duality equation gives

$$\frac{1}{f} \square f = 0. \quad (2.8)$$

The smooth solutions to this equation are topologically trivial. A multi-instanton configuration of winding number n is given by a singular solution of the form

$$f = 1 + \sum_{i=1}^n \frac{\rho_i^2}{(x - x_i)^2} \quad (2.9)$$

where ρ_i gives the scale size of the i th instanton located at x_i . Note that although f is singular, eq. (2.8) is obeyed everywhere without source terms. The corresponding gauge connection is given for $n = 1$ by

$$A_\mu = -2\rho^2 \bar{\Sigma}_{\mu\nu} \frac{x^\nu}{x^2(x^2 + \rho^2)}, \quad (2.10)$$

and is singular as $x \rightarrow 0$. To understand the solution in this singular gauge it is useful to excise a small sphere of radius ϵ surrounding the origin. On the S^3 of radius ϵ (2.10) approaches a pure gauge configuration with winding number one, but has no winding number on the S^3 at infinity. The usual smooth instanton configuration is obtained by gauge transforming (2.10) with a singular large gauge

transformation which moves the winding number from the S^3 at the origin to the S^3 at infinity. The resulting gauge connection is given by

$$A'_\mu = -2\Sigma_{\mu\nu} \frac{x^\nu}{(x^2 + \rho^2)} \quad (2.11)$$

with $\Sigma_{\mu\nu}$ the self-dual analog of $\bar{\Sigma}_{\mu\nu}$. Later on we will see that there exists a gravitational analog of this instanton solution with $e^{2\phi}$ playing the role of the scalar field f in this ansatz and with the singular behavior of (2.10) related to a semi-wormhole interpretation of the underlying geometry.

To complete the review of the gauge solution of ref. [3] we consider an E_8 or $SO(32)$ Yang–Mills instanton of scale size ρ and embed it in ten-dimensional space-time as a fivebrane solution independent of the coordinates x^a , $a = 0 \dots 5$, parallel to the worldbrane. With x^μ , $\mu = 6 \dots 9$ the transverse coordinates, we then have for the solution

$$\begin{aligned} A_\mu &= -2\Sigma_{\mu\nu} \frac{x^\nu}{(x^2 + \rho^2)} , \\ e^{2\phi} &= e^{2\phi_0} + 8\alpha' \frac{(x^2 + 2\rho^2)}{(x^2 + \rho^2)^2} , \\ H_{\mu\nu\lambda} &= -\epsilon_{\mu\nu\lambda}{}^\rho \nabla_\rho \phi , \\ g_{ab} &= \eta_{ab} , \quad g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu} . \end{aligned} \quad (2.12)$$

This configuration is a solution of the equations of motion and Ω_- is indeed an $SU(2)$ connection obeying eq. (2.6). (2.12) is annihilated by supersymmetry transformations generated by spinors ϵ_+ obeying

$$\begin{aligned} \epsilon_{\mu\nu\lambda\rho} \Gamma^{\mu\nu\lambda\rho} \epsilon_+ &= 24\epsilon_+ , \\ \epsilon_{abcdef} \Gamma^{abcdef} \epsilon_+ &= 720\epsilon_+ , \end{aligned} \quad (2.13)$$

and therefore represents a supersymmetric fivebrane soliton. A four-dimensional cross-section of this soliton is a standard Yang–Mills instanton, dressed up with the dilaton and axion.

There are two “charges” one can associate with this solution. These are the instanton winding number

$$\nu = \frac{1}{480\pi^2} \int \text{Tr } F \wedge F , \quad (2.14)$$

where the integral is over a four-dimensional cross section, and the axion charge

$$Q = -\frac{1}{2\pi^2} \int H, \quad (2.15)$$

where the integral is over an asymptotic S^3 surrounding the fivebrane. These charges are both quantized [10], with the minimal allowed values given by $\nu = 1$ and $Q = \alpha'$. For the solution (2.12) these take the values

$$\nu = 1, \quad Q = 8\alpha'. \quad (2.16)$$

Any self-dual or anti-self-dual gauge field is an equally good starting point for such a soliton. It was further shown in ref. [11] that given such a gauge field, invariance under (2.2) uniquely determines the remaining fields. The moduli space of static supersymmetric solitons is therefore identical to the moduli space $\mathcal{M}$ of a minimal winding number E_8 or $SO(32)$ instanton on $\mathbb{R}^4$. For either E_8 or $SO(32)$ this is a 120-dimensional space generated by 4 translations, 1 dilaton and 115 gauge rotations.

This solution solves the tree-level heterotic string equations of motion to lowest order in α' . As such it is a priori expected to be a valid classical solution only if the “size”, in this case ρ , is large compared to $\sqrt{\alpha'}$. In ref. [7] low-energy supersymmetry is used to argue that in fact an exact classical solution exists for every value of ρ . However, this exact solution is not well-approximated by the functional form given in eqs. (2.12).

The arguments of ref. [7] do not apply to quantum corrections, though it is conceivable that they might be extended to do so. Thus we can be confident of having a good quantum solution only if the dimensionless string coupling $g_s^2 = e^{2\phi} \ll 1$ everywhere. From (2.12) we see that this requires both $\rho \gg \sqrt{\alpha'}$ and $e^{2\phi_0} \ll 1$. Whether or not we can consistently restrict ourselves to this range of parameters depends on the eventual application we have in mind.

If we view the solution (2.12) as a soliton fivebrane then the semi-classical soliton spectrum is obtained by quantizing the collective coordinates including ρ . (2.12) would then be accurate only for the description of fivebranes whose size is large compared to $\sqrt{\alpha'}$. We can also obtain an instanton solution from (2.12) by simply taking $g_{ab} = \delta_{ab}$ rather than $g_{ab} = \eta_{ab}$. In this case we must integrate over ρ with some measure in order to calculate the instanton contribution to correlation functions. It is known that supersymmetric answers are obtained only after integrating over ρ , so we must consider configurations with small ρ , which is outside the weak-coupling regime. The integration measure for such configurations is not known, and it is not clear whether a consistent semiclassical expansion is possible. A complete understanding of how to deal with solutions at small ρ will probably require a better understanding of collective coordinates in string theory.

2.2. THE NEUTRAL SOLUTION

We now consider some interesting variations on the solution (2.12). We start by considering the solution corresponding to a size ρ instanton, and then taking the limit as the scale size ρ goes to zero. In ordinary Yang–Mills theory an instanton of size $\rho = 0$ is equivalent under singular gauge transformations to the trivial configuration. Here the situation is different. For $\rho = 0$ (2.12) is gauge equivalent to

$$\begin{aligned} A_\mu &= 0, \\ e^{2\phi} &= e^{2\phi_0} + \frac{n\alpha'}{x^2}, \\ H_{\mu\nu\lambda} &= -\epsilon_{\mu\nu\lambda}{}^\rho \nabla_\rho \phi, \\ g_{ab} &= \eta_{ab}, \quad g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu}, \end{aligned} \tag{2.17}$$

with $n = 8$, which is a non-trivial field configuration with

$$\nu = 0, \quad Q = n\alpha'. \tag{2.18}$$

Although it is only the $n = 8$ solution which is reached as a formal limit of (2.12) as $\rho \rightarrow 0$, a solution exists for any positive integer n . The solution (2.17), which we shall refer to as the neutral solution, and close relatives have been previously considered in Wick rotated form in refs. [11–15] and independently as a fivebrane in ref. [16]. The fields are apparently singular at $r = 0$, but in fact this is a coordinate artifact. The metric describes a geometry with an $S^3 \times M^6$ (where M^6 is six-dimensional Minkowski space) boundary at $r = 0$ which is infinitely far away, as can be easily seen by transforming to the coordinate $t = \ln r$. A four-dimensional transverse cross-section looks like an infinitely long semi-wormhole with one end going off to infinity. Indeed this configuration can also be reached as a limiting case of more general string wormholes described in ref. [12]. The radius of the asymptotic S^3 is equal to $\sqrt{n\alpha'}$. This means that sigma model perturbation theory is valid if $n \gg 1$, and (2.17) should then be a good approximation to an exact classical solution. However, string theory is not weakly coupled for small r (since $e^{2\phi}$ diverges there), so (2.17) cannot be trusted as a *quantum* solution far down the wormhole throat. Note that the integral of H over the S^3 is quantized in accordance with ref. [10], which also implies quantization of the radius of the wormhole throat. Physically we can think of this solution (for $n = 8$) as having a Yang–Mills instanton pushed off to the S^3 at $r = 0$.

It should be stressed that although (2.17) is in a (singular) sense a limit of (2.12), they are fundamentally different objects. Topological aspects, such as the number of fermion zero modes and the Betti numbers, differ between the two solutions, and the dynamics of the corresponding fivebranes (described in ref. [7]) can also be seen to differ. Furthermore the minimal $n = 1$ solution cannot be reached even as a singular limit of (2.12). Note also that, since the size here is quantized and there are no gauge fields, the moduli space is generated by the four bosonic translational zero modes.

The neutral configuration (2.17) involves a gravitational analog of the gauge solution (2.12) in which the self-dual SU(2) connection given by the generalized connection $\Omega_{+\mu}^{mn} = \omega_{\mu}^{mn} + H_{\mu}^{mn}$ replaces the gauge connection (2.10). (The relevance of Ω_{+} , rather than Ω_{-} , will become clear from the worldsheet analysis of sect. 3.) To see this note that the generalized connection for (2.17) is given by

$$\Omega_{+\mu}^{mn} = 2\sigma_{\mu\nu}^{mn}\nabla^{\nu}\phi, \quad (2.19)$$

with

$$\sigma_{\mu\nu}^{mn} = \delta_{\mu\nu}^{mn} - \frac{1}{2}\epsilon_{\mu\nu}^{mn} \quad (2.20)$$

anti-self-dual in both pairs of indices. The duality in (mn) is equivalent to the statement that the SO(4) connection $\Omega_{+\mu}$ is in fact an SU(2) connection. Further comparing eq. (2.19) with (2.7), we see that $\Omega_{+\mu}$ coincides with the singular gauge connection (2.10), if the instanton size ρ appearing in (2.10) is identified with $\sqrt{n\alpha'} e^{-\phi_0}$. The ϕ equation of motion (ignoring R^2 terms) is the analog of (2.8) and ensures that the Riemann tensor for the connection Ω_{+} (which is the corresponding SU(2) field strength) is self-dual.

These considerations suggest that we can construct a new type of symmetric solution by a generalization of “embedding the spin connection in the gauge group” used in Calabi–Yau compactification. In particular, we wish to embed the generalized connection Ω_{+} in an SU(2) subgroup of the gauge group so that both the gauge connection and Ω_{+} correspond to self-dual SU(2) connections. This gives yet a third distinct solution which is not smoothly connected to the neutral or gauge solutions. We now turn to the description of this solution.

2.3. THE SYMMETRIC SOLUTION

In order to embed the connection Ω_{+} in the gauge group we take the configuration of solution (2.17) augmented by a Yang–Mills instanton of scale size $\rho_n = \sqrt{n\alpha'} e^{-\phi_0}$ so that $A_{\mu} = \Omega_{+\mu}$ with A_{μ} a self-dual SU(2) connection given by

the 't Hooft ansatz (2.10)

$$\begin{aligned}
 A_\mu &= -2\rho_n^2 \bar{\Sigma}_{\mu\nu} \frac{x^\nu}{x^2(x^2 + \rho_n^2)}, \\
 e^{2\phi} &= e^{2\phi_0} + \frac{n\alpha'}{x^2}, \\
 H_{\mu\nu\lambda} &= -\epsilon_{\mu\nu\lambda}{}^\rho \nabla_\rho \phi, \\
 g_{ab} &= \eta_{ab}, \quad g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu}.
 \end{aligned} \tag{2.21}$$

This solution has

$$\nu = 1, \quad Q = n\alpha'. \tag{2.22}$$

A special limit, which we will analyze from the point of view of superconformal field theory in sect. 4, involves taking $e^{2\phi_0} \rightarrow 0$. In this limit the four-dimensional geometry is that of a cylinder whose cross section is an S^3 of constant radius. For any value of $e^{2\phi_0}$ the configuration (2.21) preserves half of the space-time supersymmetries and solves the equations of motion to lowest order in α' . The solution (2.21) corresponds to a single Yang–Mills instanton but has arbitrary axion charge n . It can be generalized to a multi-instanton solution with instanton number k using the 't Hooft ansatz (2.7) and (2.9) in which case the solution is characterized by k positive integers n_i which give both the scale size of the i th instanton and the flux of H down the i th wormhole with center at x_i .

Unlike the gauge solution (2.12) and the neutral solution (2.17) which receive corrections to higher orders in α' , we shall see there is good reason to believe that this symmetric configuration is an exact solution to heterotic string theory. One piece of evidence is provided by an explicit check of the supersymmetry variations to the next order in α' . The supersymmetry variations to order α'^3 for a supersymmetric completion of the action including R^2 , R^4 , and F^4 terms are given in ref. [17]. They find that these corrections are all linear in the tensors

$$\begin{aligned}
 T_{MNPQ} &= \text{tr } R(\Omega_+)_{[MN} R(\Omega_+)_{PQ]} - \frac{1}{30} \text{Tr } F_{[MN} F_{PQ]}, \\
 T_{MN} &= R(\Omega_+)_{MPQR} R(\Omega_+)^{PQR}{}_N - \frac{1}{30} \text{Tr } F_{MP} F_N{}^P.
 \end{aligned} \tag{2.23}$$

where $R(\Omega_+)$ is the curvature constructed from Ω_+ . Equating Ω_+ with A guarantees that these tensors vanish along with the supersymmetry variations.

Thus through third order in α' , the configuration (2.21) is supersymmetric without any corrections, while for (2.12) and (2.17) we have shown that supersymmetry can be maintained but corrections are required. In ref. [17] it is found that while the lowest-order supersymmetry transformation of the gravitino involves the connection Ω_- , the higher-order terms in the action as well as the higher-order supersymmetry transformation laws can be conveniently written in terms of the Riemann tensor constructed using the connection Ω_+ . By the same token, the heterotic correction to H is $\omega_3^L(\Omega_+) - \omega_3^{YM}$. The asymmetry between Ω_+ and Ω_- seems odd: the action seems to involve only Ω_+ , while the supersymmetry transform of ψ_μ involves Ω_- at lowest order. However, this asymmetry is essential for our solution to be consistent, and will play a crucial role when we turn to the underlying sigma model.

Of course the low-energy point of view of this section cannot establish that (2.21) is a solution to all orders in α' . In sect. 3 we will extend these arguments based on the corresponding sigma model to show that this lack of perturbative corrections persists to all orders in α' .

One can also ask about the moduli space of such solutions. If we restrict ourselves to symmetric configurations with the generalized connection embedded in the gauge group then the moduli space of solutions differs from that of the dressed Yang–Mills instantons of (2.12). Embedding the generalized spin connection in the gauge group involves choosing a map from the spin bundle into the gauge bundle. The 115 global gauge rotations of (2.21) generate moduli which deform this map. The 4 transverse translations also generate moduli of (2.21). There is *not* a modulus corresponding to the instanton size which preserves the form of the solution (2.21), since the size is proportional to the axion charge which is quantized. However there *is* a dilational modulus which does not preserve the form of (2.21) and violates the equality of the spin and gauge connection. Deformations along this modulus result in a solution which does require corrections order by order in α' . From the worldsheet point of view, they correspond to perturbations which break (4, 4) worldsheet superconformal symmetry down to (4, 0) symmetry. A similar situation occurs in Calabi–Yau compactification with the appearance of moduli which violate the equality of the spin and gauge connections and break (2, 2) superconformal symmetry down to (2, 0) symmetry [18].

One can also consider symmetric multi-instanton configurations which generalize (2.21). Since the generalized connection (2.19) is an SU(2) connection, the multi-instanton gauge field must lie in an SU(2) subgroup of E_8 or SO(32). There are $8k$ parameters characterizing a self-dual SU(2) Yang–Mills configuration. Of these $8k$ parameters, k describe the instanton sizes and $3k - 3$ describe their relative SU(2) orientations. These $4k - 3$ parameters are not free parameters of a symmetric configuration. Symmetric configurations are labelled by $4k + 3$ parameters corresponding to the locations of the k instantons and an overall SU(2), together with 112 parameters describing the embedding of SU(2) in E_8 or SO(32).

3. Sigma model arguments

To show conclusively that a given space-time configuration is a solution of string theory, we must show that it derives from an appropriate superconformal worldsheet sigma model. In this section we will study the worldsheet sigma models corresponding to the fivebranes constructed in the previous section. We will show that all of the solutions of sect. 2 possess extended worldsheet supersymmetry, of type $(4,0)$ at least, and that the symmetric fivebrane, as its name implies, has further extended $(4,4)$ supersymmetry. We will argue that, in the $(4,4)$ case, there is a non-renormalization theorem which makes the lowest-order in α' solution for the space-time fields exact. The latter issue is closely related to the question of finiteness of sigma models with torsion and with extended supersymmetry [19,20] and the results we find are slightly at variance with the conventional wisdom, at least as we understand it. We will comment upon this at the appropriate point.

First we digress to explain why we expect four-fold extended supersymmetry in this problem. The models of interest to us are structurally equivalent to a compactification of ten-dimensional space-time down to six dimensions: there are six flat dimensions (along the fivebrane) described by a free field theory and four “compactified” dimensions (transverse to the fivebrane) described by a non-trivial field theory. The fact that the “compactified” space is not compact has no bearing on the supersymmetry issue. The defining property of all the fivebranes of sect. 2 is that they are annihilated by the generators of a six-dimensional $N = 1$ *space-time* supersymmetry. In other language, they provide a compactification to six dimensions which maintains $N = 1$ space-time supersymmetry. Now, it is well-known that in compactifications to four dimensions, the sigma model describing the six compactified dimensions must possess $(2,0)$ worldsheet supersymmetry in order for the theory to possess $N = 1$ four-dimensional space-time supersymmetry [21]. Roughly speaking, the conserved $U(1)$ current of the $(2,0)$ superconformal algebra defines a free boson which is used to construct the space-time supersymmetry charges. It is also known that, if one wants to impose $N = 2$ four-dimensional space-time supersymmetry, the compactification sigma model must have $(4,0)$ supersymmetry [22]. The conserved $SU(2)$ currents of the $(4,0)$ superconformal algebra are precisely what are needed to construct the larger set of $N = 2$ spacetime supersymmetry charges. Since, by dimensional reduction, $N = 1$ supersymmetry in six dimensions is equivalent to $N = 2$ in four dimensions, the above line of argument implies that space-time supersymmetric compactifications to six dimensions (including our fivebrane) require a compactification sigma model with at least $(4,0)$ worldsheet supersymmetry. The fact that supersymmetric compactifications to six dimensions require four-fold worldsheet supersymmetry has been noted and studied elsewhere [23].

Now we turn to an explicit study of the fivebrane sigma models. The generic sigma model underlying the heterotic string describes the dynamics of D world-

sheet bosons X^M and D right-moving worldsheet fermions ψ_R^M (where D , typically ten, is the dimension of space-time) plus left-moving worldsheet fermions λ_L^a which lie in a representation of the gauge group G (typically $SO(32)$ or $E_8 \times E_8$). The generic lagrangian for this sigma model is written in terms of coupling functions G_{MN} , B_{MN} and A_M which eventually get interpreted as space-time metric, antisymmetric tensor and Yang–Mills gauge fields and has the explicit form [24]

$$\frac{1}{4\pi\alpha'} \int d^2\sigma \{ G_{MN}(X) \partial_+ X^M \partial_- X^N + 2 B_{MN}(X) \partial_+ X^M \partial_- X^N + i G_{MN} \psi_R^M \mathcal{D}_- \psi_R^N + i \delta_{ab} \lambda_L^a \mathcal{D}_+ \lambda_L^b + \frac{1}{2} (F_{MN})_{ab} \psi_R^M \psi_R^N \lambda_L^a \lambda_L^b \}, \quad (3.1)$$

where $H = dB$. In this expression, the covariant derivatives on the left-moving fermions are defined in terms of the Yang–Mills connection, while the covariant derivatives on the right-moving fermions are defined in terms of a non-riemannian connection involving the torsion (which already appeared in sect. 2)

$$\begin{aligned} \mathcal{D}_- \psi_R^A &= \partial_- \psi_R^A + \Omega_{-N}^A \partial_- X^N \psi_R^B, \\ \mathcal{D}_+ \lambda_L^a &= \partial_+ \lambda_L^a + A_N^a{}_b \partial_+ X^N \lambda_L^b. \end{aligned} \quad (3.2)$$

As in sect. 2 we use indices of type M for coordinate space indices, type A for the tangent space and type a for the gauge group. An absolutely crucial feature of this action is that the connection appearing in the covariant derivative of the right-moving fermions is the generalized connection Ω_- , *not* the Christoffel connection. This action has a naive $(1,0)$ worldsheet supersymmetry and can be written in terms of $(1,0)$ superfields. Superconformal invariance is broken by anomalies of various kinds unless the coupling functions satisfy certain “beta function” conditions [25] which are equivalent to the space-time field equations discussed in sect. 2. The dilaton enters these equations in a rather roundabout, but by now well-understood, way [24].

To proceed further, we must construct the sigma models corresponding to the various solutions studied in sect. 2 and analyze their worldsheet supersymmetries. For the generic fivebrane, (3.1) undergoes a split into a non-trivial four-dimensional theory and a free six-dimensional theory: the sigma model metric (as opposed to the canonical general relativity metric) then describes a flat six-dimensional space-time times four curved dimensions. The right-moving fermions couple via the kinetic term to the generalized connection Ω_- , which acts only on the four right-movers lying in the tangent space orthogonal to the fivebrane. The other six right-movers are free (we momentarily ignore the four-Fermi coupling) so there is a six–four split of the right-movers as well. The left-moving fermions couple to an

instanton gauge field which, as explained in sect. 2, may or may not be identified with the *other* generalized connection, Ω_+ . In all the cases of interest to us, the gauge connection is an instanton connection and acts only in some SU(2) subgroup of the full gauge group, so that four of the left-movers couple non-trivially, while the other 28 are free. Finally, the four-fermion interaction term couples together precisely those left- and right-movers which couple to the non-trivial gauge and Ω_- connections and is therefore consistent with the six–four split defined by the kinetic terms. The remaining variables can be regarded as defining a heterotic, but free, theory (6 X, 6 ψ_R and 28 λ_L) living in the six “uncompactified” dimensions along the fivebrane. From now on, we focus our attention on the non-trivial piece of (3.1) referring to the four-dimensional part of the split. For string theory consistency, it must have a central charge of 6, which would be trivially true if the connections were all flat, but is far from obvious for a non-trivial fivebrane.

We now turn to the question of worldsheet supersymmetry. The conditions for an action of the form of (3.1) to have (4,0) extended supersymmetry have been studied by many authors, but are particularly clearly stated in a paper by Howe and Papadopoulos [19]: first, there must exist three complex structures J_i (where i labels the different complex structures) which are covariantly constant with respect to the generalized connection Ω_- and which satisfy the Clifford algebra

$$J_i J_j = -\delta_{ij} + \epsilon_{ij}^k J_k. \quad (3.3)$$

Second, the gauge field strength tensor must be self-dual (this is the specialization to an SO(4) connection of the more general condition of ref. [19]). The self-duality of the gauge field strength is obviously met by all the fivebranes discussed in sect. 2. In ref. [11] it was shown that any four-dimensional supersymmetric configuration (including those with torsion) admits an H -covariantly constant complex structure, as well as a holomorphic two-form. It is then easy to show that two additional H -covariantly constant complex structures can be formed from the holomorphic two-form and its complex conjugate (with an index raised). Therefore all the configurations of sect. 2 have (4,0) supersymmetry.

The three complex structures can be explicitly constructed as follows. In sect. 2 we showed that, for the general fivebrane ansatz, the $\Omega_{\pm}$ connections have SU(2) (as opposed to SO(4)) holonomy. More particularly, we showed that (in the obvious coordinates) the two connections simply annihilate spinors of the appropriate four-dimensional chirality

$$\Omega_{\pm} \epsilon_{\mp} = 0, \quad \gamma_5 \epsilon_{\pm} = \pm \epsilon_{\pm}, \quad (3.4)$$

with γ_5 the obvious reduction to four dimensions of the ten-dimensional gamma matrix appearing in eqs. (2.13). This follows from eqs. (2.19) and (2.20), independent of any equation satisfied by the scalar field ϕ defining an ansatz such as

(2.12). The spinors covariantly constant with respect to $\Omega_{\pm}$ are then simply constant spinors of definite chirality. In the representation in which γ_5 is diagonal we have

$$\epsilon_+ = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}, \quad \epsilon_- = \begin{pmatrix} 0 \\ \chi_- \end{pmatrix}, \quad (3.5)$$

where the $\chi_{\pm}$ are arbitrary Pauli spinors. We take the χ to be unit normalized so that the ϵ are as well. χ is obviously not unique, and a convenient set of independent choices, χ_i (with $i = 1, 2, 3$), corresponds to polarization of σ along three independent axes

$$\chi_i^\dagger \sigma_j \chi_i = \delta_{ij}. \quad (3.6)$$

But given a unit-normalized constant spinor, one can construct a complex structure by the prescription

$$J_\mu^\nu = i \epsilon^\dagger \gamma_\mu^\nu \epsilon. \quad (3.7)$$

This obviously obeys the Nijenhuis integrability condition (since it is constant), and Fierz identities imply that it squares to minus one. There are three such objects J_i one can construct by using the three independent choices of χ_i and one can easily show that they satisfy (3.3). This construction works for both ϵ_+ and ϵ_- , so we can construct two triplets of complex structures $J_{\pm i}$, covariantly constant with respect to $\Omega_{\mp}$ and both satisfying (3.3). As it happens, we can also show that all the J_{+i} commute with the J_{-i} , a fact which will be useful when we discuss (4, 4) supersymmetry.

So, the sigma model built on the space-time fields of the general fivebrane automatically satisfies the conditions for (4, 0) worldsheet supersymmetry. The authors of ref. [19] also assert (with a caveat) that sigma models with this much extended supersymmetry must actually be finite (by contrast, (2, 0) supersymmetry is not strong enough to eliminate divergences). If true, this would imply that there are no higher-order in α' corrections to the space-time fields (metric, dilaton, etc.) of the fivebrane and that the central charge would have its desired free-field value of six (in short, that all the information about these solutions is contained in the classical approximation!). However, this cannot be true, since the discussion in sect. 2 makes clear that the space-time fields of the general non-symmetric fivebrane do indeed get higher-order in α' corrections, precisely in order to maintain invariance under space-time supersymmetries. These corrections signal higher-order terms in the beta functions which can only arise from higher-loop divergences in the sigma model. The loophole in the finiteness argument of Howe and Papadopoulos (recognized by them) is provided by the chiral anomalies present in a left-right asymmetric sigma model such as (3.1). For instance, the condition of covariant constancy of a complex structure involves the gauge-

invariant H (through Ω_-) and, whenever there are anomalies, the relation between H and the sigma model couplings B and A varies from order to order in α' . This can only be the case if the model has non-trivial radiative corrections, despite its $(4,0)$ supersymmetry. Although we have not worked out the details from the worldsheet point of view, in the left–right asymmetric fivebrane sigma models it is possible, by systematic modification of the space-time fields, to maintain $(4,0)$ superconformal supersymmetry. This follows from the low-energy space-time supersymmetry arguments given in ref. [7].

The above uncertainties and qualifications presumably do not afflict left–right symmetric, non-anomalous sigma models (whose extended supersymmetry would necessarily be $(4,4)$). This finally brings us to the symmetric fivebrane introduced in sect. 2. We asserted there, on the basis of space-time supersymmetry arguments, that its fields should receive no higher-order in α' corrections. Here we show that the underlying sigma model is left–right symmetric (and therefore non-anomalous) so that the finiteness argument of ref. [19] is applicable. The symmetric fivebrane is constructed by identifying the gauge connection with the “other” generalized connection Ω_+ and making that connection self-dual by imposing the condition $\square\phi = 0$ on the metric conformal factor. The result of this is that the four bosonic coordinates transverse to the fivebrane and the four non-trivially-coupled left- and right-moving fermions are governed by the worldsheet action

$$\frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ G_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu + 2B_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu \right. \\ \left. + iG_{\mu\nu} \psi_R^\mu \mathcal{D}_- \psi_R^\nu + iG_{\mu\nu} \lambda_L^\mu \mathcal{D}_+ \lambda_L^\nu + \frac{1}{2} R(\Omega_+)_{\mu\nu\lambda\rho} \psi_R^\mu \psi_R^\nu \lambda_L^\lambda \lambda_L^\rho \right\}, \quad (3.8)$$

where $\mathcal{D}_\pm$ are the covariant derivatives built out of the generalized connections $\Omega_\pm$. To exchange the roles of ψ_R and λ_L one simply replaces the curvature of Ω_- by that of Ω_+ . This exchange symmetry property relies on the non-riemannian relation

$$R(\Omega_+)_{\mu\nu\lambda\rho} = R(\Omega_-)_{\lambda\rho\mu\nu}, \quad (3.9)$$

which indeed holds for connections constructed according to eq. (2.19) when $dH = 0$. Despite the apparent asymmetry of the coupling of λ_L to Ω_+ and ψ_R to Ω_- , the theory nonetheless has an overall left–right symmetry under which $B \rightarrow -B$, and is non-anomalous. According to ref. [19], the condition for $(4,4)$ supersymmetry is that there should be two triplets $J_{\pm a}$ of complex structures, covariantly constant with respect to $\Omega_\mp$ respectively, and such that all J_+ commute with all J_- . But we showed in the discussion following (3.7) that in the situation described by (3.8), the requisite set of complex structures exists and can be explicitly constructed quite trivially. In this situation, it seems quite plausible

that the non-renormalization theorem of ref. [19] is correct, so we assert that the symmetric fivebrane has $(4, 4)$ supersymmetry, that the classical solution for its space-time fields is exact and, presumably, the central charge of the model is unrenormalized from six.

The proof given in ref. [19] involves an analysis of superfield α' perturbation theory, and as such has no bearing on the question of nonperturbative finiteness or corrections. A much stronger statement can be made, following the original argument of ref. [26], if one is willing to simply *assume* that $N = 4$ supersymmetry is not anomalous, either perturbatively or non-perturbatively. In ref. [27] this, together with the assumption that the $SU(2)$ symmetry is not anomalous, was shown to imply finiteness of $N = 4$ models without torsion. Their arguments can probably be easily extended to include torsion, but we instead argue as follows. In appendix A we prove that $N = 4$ supersymmetry (with torsion) implies that the sigma model metric is conformal to a Ricci-flat Kähler metric, and that the conformal factor (which is essentially the dilaton) solves Laplace's equation with possible singularities. (This result assumes that the torsion obeys $dH = 0$, a relation which is known to be modified by anomalies in left-right asymmetric models.) Such geometries have a finite number of moduli given by the complex structure and Kähler class of the Ricci-flat Kähler metric, as well as the relative positions and strengths of the dilaton singularities. For a single fivebrane, the relevant Ricci-flat Kähler metric is the flat metric on $\mathbb{R}^4$ and has no moduli. The strength of the dilaton singularity is related by $N = 4$ supersymmetry to the integral of H over the three-sphere around the singularity, which is quantized [10]. Therefore there are no moduli at all for the sigma model of a single symmetric fivebrane. Since there are no moduli, there are no possible counterterms which can be added to the action without violating our assumption that $N = 4$ supersymmetry can be maintained. Therefore the theory must be completely finite and conformally invariant with no corrections, either perturbative or non-perturbative in α' .

In the next section we will present some purely algebraic confirmation of these ringing claims.

4. Wess–Zumino–Witten wormhole

It is one thing to show that a sigma model is a superconformal field theory, as we have done in the previous section, and quite another to be able to classify its primary field content and calculate n -point functions of its vertex operators. Indeed, in order to answer all the interesting questions about string solitons, it would be desirable to have as detailed an algebraic understanding of the underlying conformal field theory as we already have for, say, the minimal models. We are far from having such an understanding, but in this section we will see that useful insight can be gained by studying a special limit which emphasizes the semi-wormhole throat.

Recall from sect. 2 that the (four-dimensional part of the) metric of the symmetric solution has the form

$$ds^2 = e^{2\phi} dx^2, \quad (4.1)$$

where dx^2 is the flat metric on $\mathbb{R}^4$ and

$$e^{2\phi(x)} = e^{2\phi_0} + \sum_{i=1}^n \frac{Q_i}{(x - x_i)^2}. \quad (4.2)$$

The singularities in $e^{2\phi}$ are associated with the semi-wormholes. Taking $n = 1$ and the limit $e^{2\phi_0} \rightarrow 0$ gives

$$e^{2\phi} = Q/x^2, \quad (4.3)$$

which is the solution corresponding to the wormhole throat itself. Using spherical coordinates centered on the singularity, and defining a logarithmic radial coordinate by $t = \sqrt{Q} \ln \sqrt{x^2/Q}$, the metric, dilaton and axion field strength of the throat may be written in the form

$$\begin{aligned} ds^2 &= dt^2 + Q d\Omega_3^2, \\ \phi &= -t/\sqrt{Q}, \quad H = -Q\epsilon, \end{aligned} \quad (4.4)$$

where $d\Omega_3^2$ is the line element and ϵ the volume form of the unit 3-sphere obeying $\int \epsilon = 2\pi^2$. The geometry of the wormhole is thus a 3-sphere of radius $\sqrt{Q}$ times the open line $\mathbb{R}^1$ and the dilaton is linear in the coordinate of the $\mathbb{R}^1$. Remarkably, these metric and antisymmetric tensor fields are such that the curvatures constructed from the generalized connections, defined in eqs. (2.19), are identically zero, reflecting the parallelizability of S^3 . The axion charge Q is integrally quantized. So, since Q appears in the metric, the radius of the S^3 is quantized as well.

The sigma model defined by these background fields is an interesting variant of the Wess–Zumino–Witten model and the underlying conformal field theory can, it turns out, be analyzed in complete detail. The basic observation along these lines was made in ref. [28] in the lorentzian context and euclideanized in refs. [13, 29]: the S^3 and the antisymmetric tensor field are equivalent to the $O(3)$ Wess–Zumino–Witten model of level

$$k = Q/\alpha', \quad (4.5)$$

while the $\mathbb{R}^1$ and the linear dilaton define a Feigin–Fuks-like free field theory with a background charge induced by the linear dilaton. Both systems are conformal

field theories of known central charges

$$c_{\text{wzw}} = \frac{3k}{k+2}, \quad c_{\text{ff}} = 1 + \frac{6}{k}. \quad (4.6)$$

The shift of the $\mathbb{R}^1$ central charge away from unity is a familiar background charge effect which has been exploited in constructions of the minimal models [30] and in cosmological solutions [28].

For the combined theory to make sense, the net central charge must be four. Let us for the moment consider the bosonic string. If we expand c_{wzw} in powers of k^{-1} (this corresponds to the usual perturbative expansion in powers of α'), we see instead that

$$c_{\text{tot}} = c_{\text{wzw}} + c_{\text{ff}} = 4 + O(k^{-2} \sim \alpha'^2). \quad (4.7)$$

But, we should not have expected to do any better: the field equations we solved in sect. 2 to get this solution are only the leading-order in α' approximation to the full bosonic string theory field equations and we must expect higher-order corrections to the fields and central charges. In fact, this issue can be studied in detail and it can be shown [31] that the metric and antisymmetric tensor fields are not modified and that the only modification of the dilaton is to adjust the background charge of the $\mathbb{R}^1$ (i.e. the coefficient of the linear term in ϕ) so as to maintain c_{tot} exactly equal to four.

While this is quite interesting, we are really interested in the superstring case. The leading-order in α' metric, dilaton etc. fields are the same as in the bosonic case (and because of the non-renormalization theorems we expect no corrections to them) but various fermionic terms are added to the previous purely bosonic sigma model. The structure is that of the (1,1) worldsheet supersymmetric sigma model (3.8) discussed in sect. 3. There is still an $S^3 \times \mathbb{R}^1$ split, but the component theories are supersymmetrized versions of Wess–Zumino–Witten and Feigin–Fuks. The Feigin–Fuks theory is still essentially free. In the supersymmetric WZW theory, the four-Fermi terms vanish identically because, as pointed out above, the generalized curvature vanishes for this background. As a consequence, the generalized connections are locally pure gauge and can be eliminated from the fermion kinetic terms by a gauge rotation of the same field. Since the fermions are effectively free, they make a trivial addition to the central charges of both the S^3 and the $\mathbb{R}^1$ models

$$c_{\text{wzw}} = \frac{3k}{k+2} + \frac{3}{2}, \quad c_{\text{ff}} = 1 + \frac{6}{k} + \frac{1}{2}. \quad (4.8)$$

There is, however, a small subtlety: the gauge rotation which decouples the fermions is *chiral*, and therefore anomalous, because the left- and right-moving fermions couple to two *different* pure gauge generalized connections, Ω_+ and Ω_- . The entire effect of this anomaly on the central charge turns out to be the replacement in c_{wzw} of k by $k - 2$ (the details can be found in ref. [32]) with the result that

$$c_{\text{wzw}} = \frac{3(k-2)}{k} + \frac{3}{2}, \quad c_{\text{tot}} = c_{\text{wzw}} + c_{\text{ff}} = 6. \quad (4.9)$$

Six is, of course, exactly the value we want for the central charge. The remarkable fact is that, in the supersymmetric theory, the expansion of c_{wzw} in powers of k^{-1} *terminates* at first non-trivial order and no modification of the dilaton field is needed to maintain the desired central charge of six. These results are expected from the non-renormalization theorems discussed in sects. 2 and 3, and since they derive from concrete, exactly-solved, conformal field theories, they lend plausibility to the assumptions made in the earlier general discussion. On the other hand, since the present discussion makes no reference to the (4, 4) supersymmetry which was crucial in proving the non-renormalization theorems of sect. 3, an important element is still missing.

An important clue in understanding the structure of the (4, 4) superconformal symmetry is provided by the presence of the SU(2) rotational symmetry of the symmetric solution with Yang–Mills charge one and the corresponding SU(2) Kac–Moody algebra with central charge related to the axion charge. The standard $N = 4$ superalgebra contains the energy-momentum tensor $T(z)$, four supercurrents $G^a(z)$ and three currents $J^i(z)$ of conformal weight 1, which generate an SU(2) Kac–Moody algebra. However the SU(2) in the $N = 4$ algebra must exist for a solution of arbitrary Yang–Mills winding number and its level is tied to the conformal anomaly (in our case this implies level one). It is thus distinct from the SU(2) Kac–Moody algebra described above. Thus in the special case of the single wormhole there must exist both an SU(2) Kac–Moody algebra tied to the $N = 4$ superconformal algebra and a second SU(2) Kac–Moody algebra, this time of general level n , arising from the Wess–Zumino–Witten part of the theory. Since the superconformal algebra is quite tightly constrained, it is not a priori obvious that such an $\text{SU}(2) \otimes \text{SU}(2)$ Kac–Moody is compatible with $N = 4$ supersymmetry and useful information, such as restrictions on allowed values of the central charge, might be obtained by explicitly constructing the algebra (assuming a consistent one to exist). Quite remarkably, precisely the algebra we need has already been constructed by Sevrin et al. [33], who discovered an alternate $N = 4$ superalgebra, containing an $\text{SU}(2) \otimes \text{SU}(2) \otimes \text{U}(1)$ Kac–Moody algebra, which had been missed in previous attempts at a general classification of extended superalgebras. This explicit construction in addition to establishing the presence of a (4, 4)

superconformal symmetry is a useful starting point for studying the correspondence between the instanton moduli space and perturbations of the superconformal field theory. We will return in a future publication to a more thorough exploration of this topic [34].

5. Conclusion

In this paper we have shown that soliton (or instanton) solutions of heterotic string theory exist which can be constructed rather explicitly as superconformal field theories. In the special case of (4,4) superconformal symmetry the solutions are exact solutions to string theory without higher order corrections in α' . These solutions have an enhanced SU(2) symmetry. As in the construction of special Calabi–Yau solutions in terms of orbifolds [35] or $N=2$ tensor products [36] we hope that this construction will provide a starting point for a more detailed understanding of semi-classical solutions of string theory.

It is clearly of great importance to further pursue the connection between the space-time and worldsheet points of view for these solutions. In particular it is important to construct the (super) moduli space of such solutions directly in terms of $N=4$ superconformal field theory. We will return to these issues in a later paper [34]. Eventually we would hope to have enough explicit information about this moduli space to address the issues of integration over or quantization of collective coordinates discussed in the introduction.

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Appendix A

CONSTRAINTS FROM $N=4$ SUPERSYMMETRY

In this appendix we show that $N=4$ supersymmetry (in four target space dimensions) with torsion constrains the target space metric up to a finite number of moduli. Related observations are made in ref. [6]. We assume that, by virtue of $N=4$ supersymmetry, there exist three complex structures J_{iM}^N $i=1,2,3$ satisfying the conditions [6, 19, 20]

$$\begin{aligned} \nabla_M J_{iN}^P - H_{SM}^P J_{iN}^S - H_{MN}^S J_{iS}^P &= 0, \\ J_{iM}^N J_{jN}^P &= -\delta_{ij} \delta_M^P + \epsilon_{ij}^k J_{kM}^P, \\ J_{iM}^N J_{iP}^Q g_{NQ} &= g_{MP} \quad (\text{no sum on } i), \end{aligned} \tag{A.1}$$

where H is a torsion which satisfies $dH = 0$ and ∇ is the usual covariant derivative. The relation $dH = 0$ can be modified by anomalies, so this appendix applies only to left-right symmetric theories. We will then show that the metric g must satisfy

$$g = \Omega \hat{g}, \quad \hat{\square} \Omega = 0, \quad (\text{A.2})$$

where $\hat{g}$ is a Ricci-flat Kähler metric and $\hat{\square}$ is the Laplacian with respect to $\hat{g}$.

To make the argument, define $J = J^1$ and choose complex coordinates such that

$$J_a^b = i\delta_a^b, \quad J_{\bar{a}}^{\bar{b}} = i\delta_{\bar{a}}^{\bar{b}}. \quad (\text{A.3})$$

Further, define $K = J^2 - iJ^3$ and $\bar{K} = J^2 + iJ^3$. Then (A.1) imply that, in the usual notation for tensors in complex coordinates, g is of type (1,1), K is of type (2,0), $\bar{K}$ is of type (0,2) and H is of type (1,2) + (2,1). Then the condition (A.1) on K can be written in complex notation as

$$\nabla_{\bar{a}} K_{bc} = H_{\bar{a}b}^d K_{dc} - H_{\bar{a}c}^d K_{db}. \quad (\text{A.4})$$

Contracting with $\bar{K}^{bc}$ gives

$$\bar{K}^{bc} \nabla_{\bar{a}} K_{bc} = 2H_{\bar{a}b\bar{d}} g^{b\bar{d}} = -2\nabla_{\bar{a}} \phi, \quad (\text{A.5})$$

where $\phi = \ln(\det K)$. Now in two complex dimensions, eq. (A.5) can be solved for H by dualizing to yield

$$H_{\bar{a}b\bar{c}} = -2g_{a[\bar{b}} \nabla_{\bar{c}]} \phi. \quad (\text{A.6})$$

The condition $dH = 0$ then implies that $\square \phi = 0$.

At the same time (A.1) for J^1 implies that the connection $\Omega_+ = \Gamma + H$ commutes with J^1 and so is a $U(2)$ connection. In complex coordinates

$$\Omega_{b\bar{c}}^a = \Omega_{bc}^{\bar{a}} = 0. \quad (\text{A.7})$$

This implies that

$$H_{\bar{a}b\bar{c}} = -\Gamma_{\bar{a}b\bar{c}} = -\frac{1}{2}g_{\bar{a}b,\bar{c}} + \frac{1}{2}g_{b\bar{c},\bar{a}}. \quad (\text{A.8})$$

Note that if the torsion were zero, this would imply the usual result that the metric is Kähler.

Let us now define a new metric by $\hat{g} = e^{2\phi} g$. Putting together eqs. (A.6) and (A.8) we find that

$$\hat{g}_{a[\bar{b},\bar{c}]} = e^{2\phi} H_{\bar{a}b\bar{c}} + 2e^{2\phi} g_{a[\bar{b}} \nabla_{\bar{c}]} \phi = 0, \quad (\text{A.9})$$

which is to say that $\hat{g}$ is actually Kähler.

We now want to go further and show that it is also Ricci flat, i.e. that $\hat{F}_{ab}^b = 0$. The fact that the Ω_+ connection annihilates J means that it is a $U(2) = U(1) \times SU(2)$ connection. The existence of two additional Ω_+ -invariant complex structures is easily seen to imply that the $U(1)$ part of the connection (generated by J) vanishes so that Ω_+ obeys

$$\begin{aligned} 0 &= \Omega_{ab}^b - \Omega_{a\bar{b}}^{\bar{b}} \\ &= \Gamma_{ab}^b - \Gamma_{a\bar{b}}^{\bar{b}} + H_{ab}^b - H_{a\bar{b}}^{\bar{b}} \\ &= \Gamma_{ab}^b - \Gamma_{a\bar{b}}^{\bar{b}} + 2\partial_a \phi. \end{aligned} \quad (\text{A.10})$$

This then implies that

$$\begin{aligned} \hat{F}_{ab}^b &= \hat{\Gamma}_{ab}^b - \hat{\Gamma}_{a\bar{b}}^{\bar{b}} \\ &= \Gamma_{ab}^b - \Gamma_{a\bar{b}}^{\bar{b}} + g^{b\bar{c}}(g_{\bar{c}a}\partial_b \phi + g_{\bar{c}b}\partial_a \phi) - g^{\bar{b}c}(g_{c\bar{b}}\partial_a \phi - g_{a\bar{b}}\partial_c \phi) \\ &= \Gamma_{ab}^b - \Gamma_{a\bar{b}}^{\bar{b}} + 2\partial_a \phi = 0, \end{aligned} \quad (\text{A.11})$$

which is indeed the condition that $\hat{g}$ be Ricci flat.

In summary, the conditions (A.1) for $N=4$ worldsheet supersymmetry imply that the sigma model metric is conformal to a Ricci-flat Kähler metric. The conformal factor is further restricted to obey Laplace's equation with respect to the Ricci-flat Kähler metric.

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