

ANOMALIES AND FERMION ZERO MODES ON STRINGS AND DOMAIN WALLS

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We show that the mathematical relation between non-abelian anomalies in $2n$ dimensions, the parity anomaly in $2n+1$ dimensions, and the Dirac index density in $2n+2$ dimensions can be understood in terms of the physics of fermion zero modes on strings and domain walls. We show that the Dirac equation possesses chiral zero modes in the presence of strings in $2n+2$ dimensions (such as occur in axion theories) or domain walls in $2n+1$ dimensions. We show that the anomalies due to the chiral zero modes are exactly cancelled by anomalies due to the coupling of axion-like fields to the Dirac index density or by anomalies due to the induced topological mass term.

Anomalies arise when the quantum-mechanical vacuum functional of a field theory fails to have all the symmetries of the classical lagrangian from which it is derived. If the symmetry in question is a gauge symmetry associated with a dynamical gauge field, the anomaly is intolerable and must be eliminated. The usual method is direct cancellation by the addition of extra fermion species with appropriate quantum numbers. In this paper, we will show that in certain contexts involving defects (i.e. strings and domain walls), anomalies arise which may be compensated by quantum number flows in the higher-dimensional space in which the defect is embedded, rather than cancelled. This phenomenon gives a particularly simple physical interpretation of the recently discovered mathematical connection between chiral anomalies in $2n+2$ dimensions, parity anomalies in $2n+1$ dimensions and gravitational and non-abelian anomalies in $2n$ dimensions [1-5]. It also suggests some interesting phenomenological possibilities for the axion strings which may have played a role in the development of density perturbations in the early universe [6].

Our first step will be to show that the Dirac equation for a fermion coupled to a scalar field with either a string or a domain wall structure has chiral zero modes*.

We first consider a Dirac fermion in $2n+2$ dimensions in the field of an axion string. Fermion zero modes in the presence of gauge strings have been analyzed previously in refs. [7-10]. Their existence is guaranteed by an index theorem analyzed

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* This was noted independently by R. Rhom for the case of axion strings.

in ref. [9]. We consider a complex scalar field $\Phi = \Phi_1 + i\Phi_2$ with a non-zero vacuum expectation value ν coupled to Dirac fermions in $2n$ dimensions via

$$L = \bar{\psi}(\Phi_1 + i\bar{I}\Phi_2)\psi, \quad (1)$$

where $\bar{I} = i^n \Gamma^0 \dots \Gamma^{2n+1}$ and the Γ^m are the gamma matrices in $2n+2$ dimensions. The dimensionless axion field θ is associated with the phase of Φ . A string configuration is given by $\Phi = f(\rho) e^{i\phi}$ where $x_{2n} = \rho \cos \phi$, $x_{2n+1} = \rho \sin \phi$, and $f(\rho)$ approaches zero at the origin and ν at infinity. It is convenient to let $\not{\partial}^{\text{int}} = i\bar{I}_a \partial^a$, $a = 0 \dots 2n-1$ and set $\bar{I} = I^{\text{int}} I^{\text{ext}}$ where $I^{\text{ext}} = -iI^{2n} \Gamma^{2n+1}$ and $I^{\text{int}} = i^{n+1} \Gamma^0 \Gamma^1 \dots \Gamma^{2n-1}$. I^{int} measures the chirality of modes on the string and is correlated with I^{ext} for fixed $\bar{I}$. We write ψ in terms of eigenfunctions $\psi_{\pm}$ of $\bar{I}$ and look for solutions independent of ϕ . The Dirac equation then reads

$$\begin{aligned} i\not{\partial}^{\text{int}} \psi_- + iI^{2n} (\cos \phi + iI^{\text{ext}} \sin \phi) \partial_{\rho} \psi_- &= f(\rho) e^{i\phi} \psi_+, \\ i\not{\partial}^{\text{int}} \psi_+ + iI^{2n} (\cos \phi + iI^{\text{ext}} \sin \phi) \partial_{\rho} \psi_+ &= f(\rho) e^{-i\phi} \psi_-. \end{aligned} \quad (2)$$

The solution is

$$\psi_- = \eta(x^{\text{int}}) \exp \left[- \int_0^{\rho} f(\sigma) d\sigma \right], \quad \psi_+ = -iI^{2n} \psi_-, \quad (3)$$

with $i\not{\partial}^{\text{int}} \eta = 0$ and $I^{\text{int}} \eta = -\eta$ so that the zero modes are solutions of definite chirality of the massless Dirac equation on the string. For four space-time dimensions the zero mode corresponds to a fermion traveling in one direction along the string at the speed of light with the direction of motion determined by the axial charge of the fermion.

We will also want to consider the Dirac equation in $2n+1$ dimensions in the background field of a domain wall. Fermion zero modes for domain walls have been discussed in ref. [11]. The lagrangian is

$$L = \bar{\psi} i \not{\partial} \psi + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \kappa \phi \bar{\psi} \psi - V(\phi), \quad (4)$$

where κ is a dimensional parameter. In odd dimensions the fermion mass term changes sign under parity, so ϕ is a pseudoscalar and the lagrangian is parity invariant if $V(-\phi) = V(\phi)$. If $\langle \phi \rangle = \nu$ in vacuum then the classical equations of motion admit a domain wall solution $\phi_0(x^{2n})$ similar to the kink in $1+1$ dimensions so that $\phi_0(-\infty) = -\nu$, $\phi_0(+\infty) = +\nu$. Again we set $\not{\partial}^{\text{int}} = \Gamma_a \partial^a$, $a = 0 \dots 2n-1$ and $I^{\text{int}} = i^{n+1} \Gamma^0 \dots \Gamma^{2n-1}$. We can set $I^{2n} = iI^{\text{int}}$ so that the Dirac equation reads

$$i\not{\partial}^{\text{int}} \psi - I^{\text{int}} \partial_{2n} \psi = \kappa \phi_0(x^{2n}) \psi. \quad (5)$$

The zero mode is then given by

$$\psi = \eta(x^{\text{int}}) \exp \left[- \int_0^{x^{2n}} \kappa \phi_0(y) dy \right], \quad (6)$$

with $i\not{\partial}^{\text{int}} \eta = 0$ and $I^{\text{int}} \eta = \eta$ so that the zero modes are solutions of the massless Dirac equation on the domain wall of a definite chirality.

The problem we want to consider stems from the fact that since the zero modes just displayed have a definite chirality, their coupling to gauge fields or gravity will have anomalies. For concreteness, let us consider a Dirac fermion in four space-time dimensions coupled to an axion string and having as well some sort of gauge charge. This theory is anomaly free and has local conservation of gauge charge and energy-momentum. The string zero modes are summarized by a two-dimensional chiral Fermi field (excitations travel in one direction along the string at the speed of light). The coupling of such a field to gauge or gravitational fields has the anomalies [3, 12]

$$D_a \Theta^a_b = \frac{1}{48\pi} \partial_c \partial_d \Gamma^c_{ab} \varepsilon^{ad},$$

$$D^a J_a = \frac{1}{2\pi} \varepsilon^{ab} \partial_a A_b, \quad a \cdots d = 0, 1, \quad (7)$$

where J_a is the gauge current, Θ_{ab} is the energy-momentum tensor, A_a is the background gauge field strength and Γ^a_{bc} is the Levi-Civita connection of the background gravitational field. The anomalies mean that the charge and energy-momentum carried by the string zero modes are not conserved. In order for the full theory to have locally conserved quantities, the string/fermion system must be able to exchange quantum numbers with the outside world. This exchange cannot happen via any straightforward emission or absorption of fermions because the fermions off the string have a mass (due to the scalar field vacuum expectation value) which can be made as large as we like.

The problem can usefully be rephrased in the following way: the anomalies in eq. (7) tell us that the part of the fermion determinant coming from the string zero modes is not invariant under gauge or general coordinate transformations. To be precise

$$\delta_\omega (\ln \det_{ZM}) = i \int d^2x \frac{1}{2\pi} \text{tr} (\omega \varepsilon^{ab} \partial_a A_b),$$

$$\delta_\eta (\ln \det_{ZM}) = i \int d^2x \frac{1}{48\pi} \eta^b \varepsilon^{ad} \partial_c \partial_d \Gamma^c_{ab}, \quad (8)$$

where ω and η are the parameters of the infinitesimal gauge and coordinate transformations. There must be something in the rest of the fermion determinant (coming from the massive degrees of freedom which live off the string) which exactly cancels this variation and restores gauge and coordinate invariance.

We will now discuss how the string can exchange charges with the outside world. To make things simple, imagine that the fermion has a unit $U(1)$ charge coupled to an abelian gauge field A_μ . The zero mode anomaly means that, if the electric field at the string has a non-vanishing component along the string, the net charge carried by the string zero modes must be changing with time. We want to show that this is associated with an inflow of charge from the outside. We use the method of

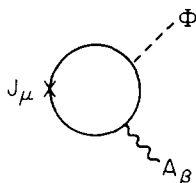


Fig. 1. One loop Feynman diagram which gives the current induced outside an axion string in adiabatic approximation.

Goldstone and Wilczek [13] to calculate the current induced in the axion field vacuum by a background electromagnetic field. In the limit of fields varying slowly compared to the inverse fermion mass, the dominant contribution to $\langle J_\mu(x) \rangle$ is given by the graph of fig. 1. It is easy to show that

$$\langle J_\mu \rangle = -i \frac{e}{16\pi^2} \varepsilon_{\mu\nu\lambda\rho} \frac{(\Phi^* \partial^\nu \Phi - \Phi \partial^\nu \Phi^*)}{|\Phi|^2} F^{\lambda\rho}, \quad (9)$$

where Φ is the axion field and $F^{\lambda\rho}$ is the background field strength. Away from the core of the string, we may write $\Phi(x) = \nu e^{i\theta(x)}$ where ν is the magnitude of the scalar vacuum expectation value and $\theta(x)$ is what is usually thought of as the axion field. In terms of θ ,

$$\langle J_\mu(x) \rangle = \frac{e}{8\pi^2} \varepsilon_{\mu\nu\lambda\rho} \partial^\nu \theta F^{\lambda\rho}. \quad (10)$$

If the string runs along the z -axis we can take $\theta(x) = \phi$ in cylindrical coordinates. If the electric field is of strength E in the string direction we see that the only non-vanishing component of the current is $J_\rho = -(e/4\pi^2\rho)E$. The net inward flux of charge is exactly what is needed to account for the charge appearing on the string via the anomaly. More formally, we can say that, because of the topology of the axion string,

$$[\partial_x, \partial_y] \theta = 2\pi \delta(x) \delta(y), \quad (11)$$

so that

$$\partial^\mu \langle J_\mu \rangle = \frac{e}{4\pi} \varepsilon^{ab} F_{ab} \delta(x) \delta(y), \quad a, b = 0, 3. \quad (12)$$

The current is conserved off the string and its divergence on the string exactly matches the anomaly. It is amusing to note that the geometry of this current flow is like that of the Hall effect: current flows in a direction perpendicular to the applied electric field.

The calculation of $J_\mu(x)$ can be converted into a determination of the effective action for the world outside the string in the presence of a background electromagnetic field:

$$L_{\text{eff}} = \frac{i}{2} \int d^4x e A^\mu \langle J_\mu \rangle = - \int d^4x \frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\lambda\rho} \partial^\mu \theta A^\nu F^{\lambda\rho}. \quad (13)$$

This expression has the slightly disturbing property of being gauge variant and we might be tempted to make it gauge invariant by integration by parts to obtain

$$L_{\text{eff}} = \frac{e^2}{32\pi^2} \int d^4x \theta \varepsilon_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho}, \quad (14)$$

the familiar axion-electromagnetism coupling. This form, although gauge invariant, is unfortunately meaningless because θ is only defined modulo 2π ! The first form of L_{eff} is well defined except for the singularity in $\partial_\mu \theta$ at $\rho = 0$. For a singular string we exclude the line $\rho = 0$ from the integral. We can easily show that the gauge variance of the first form of L_{eff} is exactly what is needed to compensate for the previously noted gauge variance of the string zero mode fermion determinant:

$$\begin{aligned} \delta_\omega L_{\text{eff}} &= -\frac{e^2}{16\pi^2} \int d^4x \varepsilon_{\mu\nu\lambda\rho} \partial^\mu \theta \partial^\nu \omega F^{\lambda\rho} \\ &= \frac{e^2}{16\pi^2} \int d^4x \varepsilon_{\mu\nu\lambda\rho} (\partial^\nu \partial^\mu \theta) F^{\lambda\rho} \omega - \frac{e^2}{16\pi^2} \int d^4x \partial^\nu (\partial_\mu \theta \varepsilon_{\mu\nu\lambda\rho} \omega F^{\lambda\rho}) \\ &= -\frac{e^2}{8\pi} \int d^4x \delta(x) \delta(y) \varepsilon^{ab} F_{ab} \omega - \frac{e^2}{16\pi^2} \lim_{\rho \rightarrow 0} \int dS^\nu \partial^\mu \theta \varepsilon_{\mu\nu\lambda\rho} \omega F^{\lambda\rho} \\ &= -\frac{e^2}{4\pi} \int d^2x \omega \varepsilon^{ab} F_{ab}. \end{aligned} \quad (15)$$

Here dS^ν is the differential surface element for a cylinder of radius ρ surrounding the string. So, both from the point of view of current conservation and gauge invariance of the functional integral, consistency is maintained.

These considerations are easily extended to include non-abelian gauge interactions and gravity. The effective lagrangian for the interaction of the axion field with a non-abelian gauge field is usually written

$$L_{\text{eff}} = \int d^4x \frac{\theta}{32\pi^2} \varepsilon_{\mu\nu\lambda\rho} \text{tr} (F^{\mu\nu} F^{\lambda\rho}), \quad (16)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ (this is just a consequence of the standard chiral anomaly). Because the 2π ambiguity in θ due to the presence of the string, this expression only makes sense if it is integrated by parts using

$$\begin{aligned} \partial^\mu K_\mu &= \varepsilon_{\mu\nu\lambda\rho} \text{tr} (F^{\mu\nu} F^{\lambda\rho}), \\ K_\mu &= 4\varepsilon_{\mu\nu\lambda\rho} \text{tr} (A^\nu \partial^\lambda A^\rho + \frac{2}{3} A^\nu A^\lambda A^\rho), \end{aligned} \quad (17)$$

to yield

$$L_{\text{eff}} = - \int d^4x \frac{1}{32\pi^2} \partial_\mu \theta K^\mu. \quad (18)$$

This result could have been obtained from diagrams analogous to fig. 1 by the

method of Goldstone and Wilczek. Under an infinitesimal gauge transformation K_μ transforms as

$$\delta_\omega K_\mu = 4\varepsilon_{\mu\nu\lambda\rho} \partial^\nu \text{tr}(\omega \partial^\lambda A^\rho), \quad (19)$$

so that L_{eff} transforms as

$$\delta_\omega L_{\text{eff}} = -\frac{1}{8\pi^2} \int d^4x \varepsilon_{\mu\nu\lambda\rho} \partial^\mu \theta \partial^\nu \text{tr}(\omega \partial^\lambda A^\rho). \quad (20)$$

As before we obtain

$$\delta_\omega L_{\text{eff}} = \frac{1}{2\pi} \int d^2x \text{tr}(\omega \varepsilon^{ab} \partial_a A_b), \quad a, b = 0, 3. \quad (21)$$

The quantity $(1/2\pi)\varepsilon^{ab}\partial_a A_b$ is the so-called consistent gauge anomaly [14, 15] in two dimensions: it describes the anomalous variation of the fermion determinant under a non-abelian gauge transformation. As a result, it will cancel against the anomaly due to the string zero modes.

These results can also be extended to treat gravitational anomalies. In 3+1 dimensions the divergence of the axial current contains a gravitational contribution [16] which implies the following coupling of the axion to gravity:

$$L_{\text{eff}} = \int d^4x \frac{\theta}{768\pi^2} \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu\alpha\beta} R_{\lambda\rho}{}^{\alpha\beta} \quad (22)$$

(where $R_{\mu\nu\alpha\beta}$ is the curvature tensor). Because of the identity

$$\begin{aligned} \partial_\mu K^\mu &= \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu\alpha\beta} R_{\lambda\rho}{}^{\alpha\beta}, \\ K^\mu &= 4\varepsilon^{\mu\nu\alpha\beta} (I^{\lambda}{}_{\nu\rho} \partial_\alpha \Gamma^{\rho}{}_{\beta\lambda} - \frac{2}{3} I^{\lambda}{}_{\nu\rho} I^{\tau}{}_{\alpha\lambda} \Gamma^{\rho}{}_{\beta\tau}). \end{aligned} \quad (23)$$

This interaction may be rewritten by integration by parts as

$$L_{\text{eff}} = -\frac{1}{768\pi^2} \int d^4x \partial_\mu \theta K^\mu. \quad (24)$$

Although the first form of the interaction is general coordinate invariant, it is incorrect because of the 2π ambiguity in θ in the presence of a string. The second form, although it varies under general coordinate transformations, is well-defined in the presence of a string and is precisely what we would obtain by the Goldstone-Wilczek type of calculation.

Just as in the gauge field case the non-invariance of this form is precisely what is needed to cancel the anomaly associated with the string zero modes: To see this, we observe that the variation of the connection under a general coordinate transformation parametrized by η_μ is

$$\delta I^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\alpha\nu} \partial_\mu \eta^\alpha + I^{\lambda}{}_{\mu\alpha} \partial_\nu \eta^\alpha - \Gamma^{\alpha}{}_{\mu\nu} \partial_\alpha \eta^\lambda + \partial_\alpha \Gamma^{\lambda}{}_{\mu\nu} \eta^\alpha + \partial_\mu \partial_\nu \eta^\lambda. \quad (25)$$

The corresponding variation of K_μ is

$$\delta K_\mu = K^\lambda \partial_\lambda \eta^\mu + \partial_\lambda K^\mu \eta^\lambda + 4\varepsilon^{\mu\nu\alpha\beta} \partial_\nu (\partial_\sigma \eta^\rho \partial_\alpha \Gamma^\sigma_{\beta\rho}). \quad (26)$$

The first two terms correspond to the way K_μ would transform if it were an ordinary tensor. The variation of eq. (24) is entirely due to the non-tensor part (NT) of K_μ :

$$\begin{aligned} \delta \int d^4x \frac{1}{768\pi^2} \partial_\mu \theta K^\mu &= \frac{1}{768\pi^2} \int d^4x \partial_\mu \theta \delta K^\mu_{NT} \\ &= \frac{1}{192\pi^2} \int d^4x \partial_\mu \theta \varepsilon^{\mu\nu\alpha\beta} \partial_\nu (\partial_\sigma \eta^\rho \partial_\alpha \Gamma^\sigma_{\beta\rho}) \\ &= \frac{1}{48\pi} \int d^2x \varepsilon^{ab} \partial_\sigma \eta^\sigma \partial_a \Gamma^\sigma_{b\rho}, \quad a, b = 0, 3, \end{aligned} \quad (27)$$

where we have used the same manipulations used to derive eq. (15). If we choose η_ρ to correspond to a reparametrization of the $(1+1)$ -dimensional space spanned by the string, we find that

$$\delta \int d^4x \frac{1}{768\pi^2} \partial_\mu \theta K^\mu = \frac{1}{48\pi} \int d^2x \varepsilon^{ab} \partial_d \eta^c \partial_a \Gamma^d_{bc}, \quad a, b = 0, 3. \quad (28)$$

This is precisely what is needed to cancel the anomaly in general coordinate invariance coming from the chiral zero modes bound to the string (compare with eq. (7)).

Since the zero mode anomaly can be interpreted as causing the energy-momentum carried by the string to change with time in the presence of an external gravitational field, we expect to be able to identify an energy-momentum tensor, $\Theta_{\mu\nu}$, which describes the inflow of that energy-momentum from the region outside the string. We determine $\Theta_{\mu\nu}$ by the usual method of computing the response of the action of eqn. (24) to a general variation of the metric. Making use of

$$\delta I^{\tau\tau}_{\nu\rho} = \frac{1}{2} g^{\tau\gamma} [(\delta g_{\gamma\nu})_{;\rho} + (\delta g_{\gamma\rho})_{;\nu} - (\delta g_{\nu\rho})_{;\gamma}], \quad (29)$$

we find

$$\begin{aligned} T^{\gamma\nu} &= \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \frac{1}{768\pi^2} \partial_\mu \theta K^\mu, \\ &= \frac{1}{384\pi^2} \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}} (\theta_{;\mu} R_{\alpha\beta}{}^{\gamma\delta})_{;\delta} + (\gamma \leftrightarrow \nu). \end{aligned} \quad (30)$$

It is not too hard to show that this energy-momentum tensor describes the desired "gravitational Hall effect" flow of energy-momentum onto the string in the presence of a background gravitational field.

We have not touched on the interesting question of the fate of the charge or energy-momentum which is deposited on the string zero modes by the process we

have described. In part, this question has been answered by Witten's work on superconducting strings [10] which was the original inspiration for our investigation. Witten replaces the zero modes by a one-dimensional Bose field (the bosonization trick) and describes their interaction with an external electromagnetic field by a simple quadratic action. This method shows the existence of a supercurrent on the string quite directly and allows the calculation of the scattering of external electromagnetic waves from the string. In our case, by analogy, we expect that charges deposited on the string will be propagated away from their point of deposition by a supercurrent. There is a crucial difference, however: since our zero modes are chiral and propagate in only one direction, they cannot be described by the usual sort of bosonization field which inevitably propagates in both directions. A simple description of the dynamics of the chiral supercurrent, its interaction with external electromagnetic fields and the associated phenomenology is surely possible, but we have yet to work it out.

Similar questions arise in the study of the interaction of strings with gravitational fields. In that case, our arguments show that energy-momentum flows onto the string and the above discussion suggests that it then propagates along the string in some kind of supercurrent. We do not yet have a detailed description of this energy-momentum supercurrent and have some reasons to expect it to be a bit peculiar. In particular, since the zero modes are chiral, their net energy and momentum must be equal (if the zero modes are right-propagating). On the other hand, the inflow energy-momentum has no need to satisfy such a restriction and it is not yet clear how the string absorbs the deficit. We hope to return to these questions in a subsequent publication.

An interesting variant of the above line of argument arises in the case of domain walls embedded in a space of one higher dimension. For simplicity we will consider a $(1+1)$ -dimensional domain wall embedded in $2+1$ dimensions. As mentioned earlier, there are chiral zero modes tied to the domain wall and an associated anomaly. Once again, there must be a gauge-variant contribution to the full action in $2+1$ dimensions which cancels the gauge variation due to the anomaly.

It is known that in $2+1$ dimensions there exist parity non-invariant topological mass terms for gauge and gravitational fields [17] and that these terms are induced by fermion loop radiative corrections even if they are not present in the classical action [18]. These induced mass terms depend on the sign of the underlying fermion's mass. If the fermion mass is induced by coupling to the scalar field of a domain wall, the sign of the mass is different on the two sides of the wall. The appropriate topological mass term for a non-abelian gauge field in the presence of a domain wall ϕ_0 is then

$$L_{\text{eff}} = \frac{1}{16\pi} \int d^3x \frac{\phi_0(x_2)}{|\phi_0(x_2)|} \epsilon^{\mu\nu\alpha} \text{tr} (A_\mu \partial_\nu A_\alpha + {}^2_3 A_\mu A_\nu A_\alpha). \quad (31)$$

(This could have been obtained directly from a one-loop calculation similar to that leading to eq. (13.)

Apart from a factor $\phi_0/|\phi_0|$, the integrand of L_{eff} is essentially the time component of the Chern–Simons current of eq. (17). This current varies under an infinitesimal gauge transformation according to eq. (19). The corresponding variation of L_{eff} is

$$\delta_\omega L_{\text{eff}} = \frac{1}{16\pi} \int d^3x \frac{\phi_0(x_2)}{|\phi_0(x_2)|} \varepsilon^{\mu\nu\sigma} \partial_\mu \text{tr}(\omega \partial_\nu A_\sigma). \quad (32)$$

As before we exclude the point $x_2 = 0$ where $\phi_0/|\phi_0|$ is ill-defined. With the approximation $\phi_0/|\phi_0| \approx \varepsilon(x_2)$. We then have

$$\begin{aligned} \delta_\omega L_{\text{eff}} &= -\frac{1}{16\pi} \int d^3x [\partial_\sigma(\varepsilon(x_2)) \varepsilon^{\mu\nu\sigma} \text{tr}[F_{\mu\nu}\omega] - \partial_\sigma(\varepsilon(x_2)) \varepsilon^{\mu\nu\sigma} \text{tr}[F_{\mu\nu}\omega]] \\ &= -\frac{1}{8\pi} \int d^3x \delta(x_2) \varepsilon^{ab} \text{tr}[F_{ab}\omega] - \frac{1}{16\pi} \lim_{\delta \rightarrow 0} (\varepsilon(\delta) - \varepsilon(-\delta)) \varepsilon^{ab} \text{tr}[F_{ab}\omega], \end{aligned} \quad (33)$$

which gives

$$\delta_\omega L_{\text{eff}} = -\frac{1}{4\pi} \int d^2x \varepsilon^{ab} \text{tr}(F_{ab}\omega). \quad (34)$$

This is precisely the $(1+1)$ -dimensional non-abelian anomaly and will cancel against the zero mode anomaly. Similar manipulations with similar results can be carried out for gravitational fields.

While our results for anomalies are not new, their physical interpretation is. Among other things, we have given a physical context for understanding the recently discovered connection between the chiral anomaly in $2n+2$ dimensions and the gauge anomaly in $2n$ dimensions: the chiral anomaly specifies the coupling between gauge fields and the axion field; if the axion field has the topology implied by the presence of a string, this coupling leads, in the presence of a gauge field, to a net flow of gauge charge (or energy-momentum) onto the string; this in turn means that there must be a gauge anomaly, i.e. a non-conservation of gauge charge, associated with the lower-dimensional string system. The domain wall version of these arguments allows us to connect gauge anomalies in even dimensions with the existence of induced parity-odd gauge mass terms in the next higher dimension (note here that the higher-dimensional theory does not have any anomaly in the usual sense). Again, in the presence of a domain wall and a gauge field, the induced mass term leads to a net influx of gauge charge onto the domain wall, implying anomalous non-conservation of gauge charge on the lower-dimensional domain wall system.

We find it interesting that theories with anomalies may be given a consistent interpretation by embedding them in higher-dimensional space-times. This interpretation is rather orthogonal to the usual view in which anomalous quantities fail to be conserved because they flow in and out of the boundaries of momentum space. These notions might be relevant to attempts to describe $(3+1)$ -dimensional space-time as a topological defect in high-dimensional space [19]. Our results will also presumably have direct application to the evolution of axion strings in the pre-QCD

phase transition era of the early universe, but we have yet to explore the details of this phenomenology.

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References

- [1] M.F. Atiyah and I.M. Singer, in *Proc. Nat. Acad. Sciences* 81 (April 1984) to appear;
B. Zumino, *Lectures at Les Houches Summer School* (1983);
R. Stora, *Cargèse Lectures* (1983);
P. H. Frampton and T. W. Kephart, *Phys. Rev. Lett.* 50 (1983) 1343, 1347 and *Phys. Rev. D* 28 (1983) 1010.
- [2] B. Zumino, W. Yong-Shi and A. Zee, *Nucl. Phys.* B239 (1984) 477
- [3] L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* B234 (1983) 269
- [4] N. Redlich, *Phys. Rev. Lett.* 52 (1984) 18;
A.J. Niemi and G. Semenoff, *Phys. Rev. Lett.* 52 (1983) 2077;
- [5] L. Alvarez-Gaumé and P. Ginsparg, *Harvard preprint HUTP-84/A016*;
O. Alvarez, I. Singer and B. Zumino, *UCB preprint PTH-84/9*
- [6] F. W. Stecker and Q. Shafi, *Phys. Rev. Lett.* 50 (1983) 928
- [7] J. Kiskis, *Phys. Rev. D* 15 (1977) 2329;
M. Anousrian, *Phys. Lett.* 70B (1977) 301;
N.K. Nielsen and B. Schroer, *Nucl. Phys.* B120 (1977) 62
- [8] R. Jackiw and P. Rossi, *Nucl. Phys.* B190 [FS3] (1981) 681
- [9] E. Weinberg, *Phys. Rev. D* 24 (1981) 2669
- [10] W. Witten, *Superconducting strings*, IAS preprint (1984)
- [11] R. Jackiw and C. Rebbi, *Phys. Rev. D* 13 (1976) 3398;
C. Callias, *Comm. Math. Phys.* 62 (1978) 213
- [12] R. Jackiw, *Lectures at the 1973 Int. School of Nucl. Phys.*, Ettore Majorana
- [13] J. Goldstone and F. Wilczek, *Phys. Rev. Lett.* 47 (1981) 986
- [14] J. Wess and B. Zumino, *Phys. Lett.* 37B (1971) 95
- [15] W. Bardeen and B. Zumino, *Nucl. Phys.* B244 (1984) 421
- [16] R. Delbourgo and A. Salam, *Phys. Lett.* 40B (1972) 381;
T. Eguchi and P. Freund, *Phys. Rev. Lett.* 37 (1976) 1251
C. Isham, S. Deser and M. Duff, *Phys. Lett.* 93B (1980) 419
- [17] W. Siegel, *Nucl. Phys.* B156 (1979) 135;
R. Jackiw and S. Templeton, *Phys. Rev. D* 23 (1981) 2291;
J. Schonfeld, *Nucl. Phys.* B185 (1981) 157;
S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* 48 (1982) 975
- [18] A.N. Redlich, *Phys. Rev. Lett.* 52 (1984) 18;
I. Affleck, J. Harvey and E. Witten, *Nucl. Phys.* B206 (1982) 413
- [19] T. Regge and C. Teitelboim, *Marcel Grossman Meeting on General Relativity*, 1975 (North-Holland, 1977) 77;
S. Deser, F. Pirani and D. Robinson, *Phys. Rev. D* 14 (1976) 3301;
K. Akama, *Gauge theory and gravitation*, *Proc. Int. Symp. at Nara, Japan* (Springer, 1983) 267;
V. Rubakov and M. Shaposhnikov, *Phys. Lett.* 125B (1983) 136;
B. Holdom, *Stanford preprints ITP-738, ITP-744* (1983); *Nucl. Phys.* B233 (1984) 413